MAT 308 Practice Final

Spring 2024

Problem 1. Find a solution $y\colon \mathbb{R} \to \mathbb{R}$ to the boundary value problem

$$y'' - 2y' + 2y = 0;\ y(0) = 0;\ y(\pi/2) = e^{\pi/2}.$$

Problem 2. Find all solutions $y\colon \mathbb{R} \to \mathbb{C}$ to the initial value problem

$$y'' + 2y' + 2y = 2; y(0) = 0.$$

Problem 3. Let $b, c \in \mathbb{R}$ and suppose $b^2 - 4c < 0$. Prove that there exists some $T \in \mathbb{R}_{\neq 0}$ and $K \in \mathbb{R}$ such that any solution $y\colon \mathbb{R} \to \mathbb{R}$ of

$$y'' + by' + cy = 0$$

satisfies

$$y(x + T) = K \cdot y(x)$$

for all $x \in \mathbb{R}$.

Problem 4. Fix some integer $N > 0$.

(a) For each $k \in \mathbb{N}$, let $u_k = e^{2\pi i k/N}$. Prove that $u_k^N = 1$ for all k, and that $u_j \neq u_k$ for all $0 \leq j < k \leq N - 1$. *Hint: For the second claim, show that $u_k/u_j \neq 1$.*

(b) Find the general solution $y\colon \mathbb{R} \to \mathbb{C}$ of the differential equation.

$$y^{(N)} = y.$$

Problem 5. Recall that the Laplace transform $\mathcal{L}[f]$ of a function $f\colon [0, \infty) \to \mathbb{R}$, if it exists, is defined by $\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt$ for all $s \gg 0$.
Prove for any differentiable function $f\colon [0, \infty) \to \mathbb{R}$, if $\mathcal{L}[f]$ and $\mathcal{L}[f']$ both exist, and if $\lim_{t\to\infty} e^{-ts} f(t) = 0$ for all $s \gg 0$, then

$$\mathcal{L}[f'](s) = -f(0) + s\mathcal{L}[f](s).$$

Problem 6. Find the general solution $x, y, z\colon \mathbb{R} \to \mathbb{R}$ of the following system.

$$\begin{aligned}\dot{x} &= 2x + y\\ \dot{y} &= 2y + z\\ \dot{z} &= 2z\end{aligned}$$

Problem 7. Let

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Find an invertible matrix $U \in \mathbb{R}^{2\times 2}$ and a diagonal matrix $D \in \mathbb{R}^{2\times 2}$ such that

$$A = UDU^{-1}.$$

Problem 8. Find a solution $x, y, z\colon \mathbb{R} \to \mathbb{R}$ to the following initial value problem.

$$\begin{aligned} \dot{x} &= 2x + 3y - 4z \\ \dot{y} &= 2x + 5y \\ \dot{z} &= x - 7y - 1z \\ x(1) &= y(1) = z(1) = 0 \end{aligned}$$

Problem 9. Find the general solution $w, x, y, z\colon \mathbb{R} \to \mathbb{R}$ to the following system.

$$\begin{aligned} \dot{w} &= x \\ \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= 1 \end{aligned}$$

Problem 10. Find the general solution $x, y\colon \mathbb{R} \to \mathbb{R}$ to the following system.

$$\begin{aligned} \dot{x} &= x + y + t \\ \dot{y} &= y + 1 \end{aligned}$$