Homework 4 (Updated Mar. 26)

Due: Wednesday, March 26

MAT 308, Spring 2025

Problem 1. Let $V = \mathbb{R}^{2\times 2}$ be the vector space of 2×2 matrices. Let $W = \{A \in \mathbb{R}^{2\times 2} \mid A^\top = A\} \subset V$ be the subspace consisting of symmetric matrices; you may take for granted that this is a subspace.

Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be the ordered bases of V and W, respectively, given by

$$\mathcal{B}_V = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) \quad \text{and} \quad \mathcal{B}_W = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right).$$

You may take for granted that these are indeed bases.

Consider the map $f\colon V \to W$ given by $f(A) = \frac{1}{2}(A + A^\top)$; you may take for granted that this is indeed linear.

Write down the matrix $F \in \mathbb{R}^{3\times 4}$ representing f with respect to the bases $\mathcal{B}_V$ and $\mathcal{B}_W$.

(This should have the property that if $\mathbf{v} \in \mathbb{R}^4$ is the coordinate vector of an element $A \in V$ with respect to $\mathcal{B}_V$, then $F \cdot \mathbf{v} \in \mathbb{R}^3$ should be the coordinate vector of $f(A)$ with respect to $\mathcal{B}_W$.)

Problem 2. For each of the following second order linear differential equations, (i) write down the corresponding characteristic equation, (ii) find its roots, (iii) use these to write down the general solution $y\colon \mathbb{R} \to \mathbb{R}$ to the differential equation, (iv) find the values of the coefficients in the general solution which solve the given initial conditions or boundary-conditions.

(i) $2y'' - 3y' + y = 0;\ y(0) = 1,\ y'(0) = 0$

(ii) $y'' - y' = 0;\ y(0) = 1,\ y(1) = 0$

(iii) $y'' + y' - 6y = 0;\ y(0) = 2,\ y'(0) = 2$

(iv) $y'' + 2y' + y = 0;\ y(1) = 1,\ y'(1) = 1$

Problem 3. Consider the differential equation $y'' + (1/x)y' - (1/x^2)y = 0$, where $y\colon (0,\infty) \to \mathbb{R}$. This can be written in terms of operators as $(D^2 + (1/x)D - 1/x^2)y = 0$.

(a) Show that the equation can be written as $D(D + 1/x)y = 0$.

(b) Solve the equation in part (a) by successively solving two first-order equations.

(c) Show that the differential operators $D(D + 1/x)$ and $(D + 1/x)D$ are not equal.

(d) Solve the equation $(D + 1/x)Dy = 0$ where $y\colon (0,\infty) \to \mathbb{R}$.

Problem 4.

(a) Show that the boundary-value problem $y'' + y = 0;\ y(0) = 1,\ y(\pi) = 1$ has no solution $y\colon \mathbb{R} \to \mathbb{R}$. *Hint: You know all the solutions to $y'' + y = 0$.*

(b) Show that the boundary-value problem $y'' + y = 0;\ y(0) = 1,\ y(2\pi) = 1$ has infinitely many solutions $y\colon \mathbb{R} \to \mathbb{R}$.

Problem 5. Consider a differential equation $y'' + ay' + by = 0$ with $a, b \in \mathbb{R}$, and suppose that its characteristic equation has two real roots $r_1, r_2 \in \mathbb{R}$. Prove that for any distinct real numbers $x_1 \neq x_2 \in \mathbb{R}$ and any $y_1, y_2 \in \mathbb{R}$, the boundary value problem

$$y'' + ay' + by = 0, \quad y(x_1) = y_1, \quad y(x_2) = y_2$$

has a unique solution $y\colon \mathbb{R} \to \mathbb{R}$. *Hint: Consider the cases $r_1 = r_2$ and $r_1 \neq r_2$ separately.*

Update Mar. 26: "real roots $r_1 = r_2$" changed to "real roots r_1, r_2" in Problem 5.