Homework 5

Due: Wednesday, April 9

MAT 308, Spring 2025

Problem 1. For each of the following complex numbers z_i: (i) express z_i in the form $re^{i\theta}$ with $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$, and (ii) find the square roots of z_i, i.e., the two distinct complex numbers w, w' with $w^2 = (w')^2 = z_i$.

(a) $z_1 = 1 + i$ (b) $z_2 = i$ (c) $z_3 = \sqrt{3} + i$ (d) $z_4 = -7$

Problem 2. Let $f \colon \mathbb{C} \to \mathbb{R}^{2\times 2}$ be the function defined by

$$f(x+iy) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

Prove the following:

(a) $f(z+w) = f(z) + f(w)$ for all $z, w \in \mathbb{C}$.

(b) $f(z \cdot w) = f(z) \cdot f(w)$ for all $z, w \in \mathbb{C}$ (where the "$\cdot$" on the right-hand side is matrix multiplication).

(c) $f(re^{i\theta}) = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for all $\theta \in \mathbb{R}$.

(d) Let $g \colon \mathbb{C} \to \mathbb{R}^2$ be the function defined by $g(x+iy) = \begin{bmatrix} x \\ y \end{bmatrix}$. (According to our definition of $\mathbb{C}$, g is actually just the identity function.) Prove that $g(z \cdot w) = f(z) \cdot g(w)$ for all $z, w \in \mathbb{C}$.

Problem 3. Recall that for a continuous function $f \colon [a,b] \to \mathbb{C}$, we define its integral $\int_a^b f(x)\,\mathrm{d}x$ component-wise: if $f(x) = u(x) + iv(x)$, we define $\int_a^b f(x)\,\mathrm{d}x = \int_a^b u(x)\,\mathrm{d}x + i\int_a^b v(x)\,\mathrm{d}x$. Equivalently (by the fundamental theorem of calculus), $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$, where $F \colon [a,b] \to \mathbb{C}$ is any anti-derivative of f (meaning $F' = f$, where this is also defined component-wise: if $F(x) = U(x) + iV(x)$, then $F'(x) = U'(x) + iV'(x)$).

(a) Compute $\int_0^{2\pi} e^{inx}\,\mathrm{d}x$ for all $n \in \mathbb{Z}$.

(b) On the vector space $\mathcal{C}^0([0,2\pi],\mathbb{C})$ of continuous functions $[0,2\pi] \to \mathbb{C}$, we define an inner product by $\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}g(x)\,\mathrm{d}x$ for continuous $f, g \colon [0,2\pi] \to \mathbb{C}$. (You may take for granted that this is an inner product). Show that $\{f_n \mid n \in \mathbb{Z}\} \subset \mathcal{C}^0([0,2\pi],\mathbb{C})$ is an orthonormal set, where $f_n \colon [0,2\pi] \to \mathbb{C}$ is given by $f_n(x) = e^{inx}$.

Problem 4. For each of the following second order linear differential equations, (i) write down the corresponding characteristic equation, (ii) find its roots, (iii) use these to write down the general real solution $y \colon \mathbb{R} \to \mathbb{R}$ to the differential equation (note that this may involve replacing complex exponentials by $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$) (iv) find the values of the coefficients in the general solution which solve the given initial conditions or boundary conditions.

(a) $y'' + 2y = 0;\ y(0) = 0,\ y'(0) = 1$

(b) $3y'' - y' + y = 0;\ y(0) = 0,\ y(1) = 0$

(c) $2y'' + y' + y = 0;\ y(0) = 4,\ y'(0) = -1$

Repeat steps (i)-(iv) for the following differential equations, but in step (iii), instead find the general *complex* solution $y \colon \mathbb{R} \to \mathbb{C}$.

(d) $y'' - 6iy' - 9y = 0;\ y(0) = -1,\ y'(0) = 0$

(e) $y'' - (1+i)y' + iy = 0;\ y(0) = 1,\ y'(0) = 1 + i$

Problem 5. Given $c_1, c_2, \beta \in \mathbb{R}$, show that the function $y(x) = c_1 \cos \beta x + c_2 \sin \beta x$ can also be expressed (i) in the form $A \cos(\beta x + \varphi)$ for some $A, \varphi \in \mathbb{R}$ or (ii) in the form $B \sin(\beta x + \theta)$ for some $B, \theta \in \mathbb{R}$.

(This gives an alternative way to express the general real solution to a real second-order differential equation whose characteristic equation has negative discriminant.)

Problem 6. Find the general (complex) solution to each of the following differential equations:

(a) $y''' + y = 0$ (b) $y^{(4)} + 2y'' + y = 0$ (c) $y^{(4)} = 0$ (d) $D(D-1)^3 y = 0$