**Homework 6**

Due: Wednesday, April 23

MAT 308, Spring 2025

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**Problem 1.** Find the general *real* solution to each of the following differential equations:

(a) $y''' + y = 0$ (b) $y^{(4)} + 2y'' + y = 0$

**Problem 2.** Let $V$ and $W$ be real vector spaces.

(a) Prove that for any linear map $f\colon V \to W$ and $\mathbf{w} \in W$, the set $\{\mathbf{v} \in V \mid f(\mathbf{v}) = \mathbf{w}\}$ is either empty, has a single element, or has infinitely many elements.

(b) Suppose $V = W = \mathbb{R}$ are both the vector space of 1-tuples over $\mathbb{R}$. Find examples of $\mathbf{w}$ and $f$ for which each of these three possibilities is realized.

**Problem 3.** Recall that a falling body with mass $m$ experiences a gravitational force $mg$ (with $g \approx 9.8\,\mathrm{m/s^2}$) and hence, in the absence of any other forces, its distance $y(t)$ from the earth satisfies $my'' = -mg$ and hence $y'' = -g$. Taking air resistance into account, we suppose that this imposes a force proportional to the velocity, and hence of the form $-ky'$ for some constant $k > 0$ (with units of force per velocity, or inverse time). Then $y$ satisfies $my'' = -ky' - mg$ or

$$y'' + \frac{k}{m}y' = -g.$$

Prove the following:

(a) Show that the general solution to this equation is $y = c_1 + c_2 e^{-kt/m} - \frac{mg}{k}t$.

(b) Show that $\lim_{t\to\infty} y'(t) = -mg/k$ for any solution $y$; this limit is called the *terminal velocity* of the falling body.

**Problem 4.** Use the method of undetermined coefficients to find the general solution $y\colon \mathbb{R} \to \mathbb{R}$ to the following inhomogeneous differential equations:

(a) $y'' - y = e^{2x}$ (b) $y'' + y = \cos x$ (c) $y'' - 2y' + y = xe^x$

**Problem 5.** Consider the linear differential operator $L = D^2 - 2xD + x^2 - 1$.

(a) Show that $L$ can be written in the form $(D - u(x))(D - v(x))$ for appropriate functions $u, v$. *Hint: You can take $u$ and $v$ to be polynomials in $x$.*

(b) Use the factorization from (a) to find the general real solution the homogeneous equation $y'' - 2xy' + x^2y - y = 0$.

(c) Using the two independent homogeneous solutions $y_1$ and $y_2$ found in (b), write down the corresponding Wronskian determinant $w(x)$ and the Green's function $G(x, t)$.

(d) Use the Green's function from (c) to find a solution $y\colon \mathbb{R} \to \mathbb{R}$ to the following equations satisfying $y(0) = y'(0) = 0$.

(i) $y'' - 2xy' + x^2y - y = e^{x^2/2}$ (ii) $y'' - 2xy' + x^2y - y = xe^{x^2/2}$

**Problem 6.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and suppose that the Laplace transform $\mathcal{L}[f]$ exists.

(a) Prove that $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s - a)$ for $a \in \mathbb{R}$.

(b) Prove that $\mathcal{L}[f(t - a)](s) = e^{-as}\mathcal{L}[f](s)$ for $a > 0$, where we assume $f(t) = 0$ for $t < 0$.

(c) Prove that $\mathcal{L}[f(bt)](s) = \frac{1}{b}\mathcal{L}[f](s/b)$ for $b \neq 0$.

(d) Prove that $\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}}$ for all $n \geq 0$. *Hint: Proceed by induction. You may want to use the formula for $\mathcal{L}[f']$ proven in class.*

(e) Prove that $\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}}$ for all $n \geq 0$. *Hint: Use (a).*

**Problem 7.** Use the Laplace transform to find a solution $y\colon [0, \infty) \to \mathbb{R}$ to the following initial-value problem:

$$y'' + 2y' + y = 1,\ y(0) = y'(0) = 0$$