**Homework 7**

Due: Friday, May 9

MAT 308, Spring 2025

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**Problem 1.** Write down a first-order system of differential equations which is equivalent to the equation

$$y^{(3)} = (\ddot{y})^2 - y\dot{y} - t.$$

**Problem 2.**

(a) Find the (real) eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

(b) Find a diagonal matrix $D$ and an invertible matrix $U$ such that $A = UDU^{-1}$. Compute the inverse $U^{-1}$.

(c) Compute the exponential $e^{tA}$ using the results from (b).

(d) Find the general solution $y \colon \mathbb{R} \to \mathbb{R}^3$ of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$.

**Problem 3.**

(a) Find the (real) eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(b) Compute the exponential $e^{tA}$ by finding functions $b_0(t)$, $b_1(t)$, and $b_1(t)$ such that $e^{tA} = b_0(t)\mathrm{I} + b_1(t)A + b_2(t)A^2$. *Hint: Recall that Theorem 13.2.4 gives conditions which determine* $b_0$, $b_1$, *and* $b_2$.

(c) Find the general solution $y \colon \mathbb{R} \to \mathbb{R}^3$ of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$.

**Problem 4.** Find the (complex) eigenvalues and eigenvectors of the rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

**Problem 5.** Let $L \in \mathcal{L}(V)$ be a linear operator on a vector space $V$. And let $L^n$ be the operator obtained by composing $L$ with itself $n$ times (i.e., $L^0 = \mathrm{id}_V$ and $L^{n+1} = L \circ L^n$ for $n \geq 0$).

Prove that if $\lambda$ is an eigenvalue of $L$, then $\lambda^n$ is an eigenvalue of $L^n$ for all $n \geq 0$.

**Problem 6.** One can show that if two matrices $A$ and $B$ commute, then $e^{A+B} = e^A \cdot e^B$. (You do not need to show this.)

Find $2 \times 2$ matrices $A$ and $B$ for which $e^{A+B}$ and $e^A \cdot e^B$ are *not* equal.

**Problem 7.** Let $V$ be a real inner product space and let $L \in \mathcal{L}(V)$ be an orthogonal transformation. (Recall that this means that $\langle L\mathbf{v}, L\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all $\mathbf{v}, \mathbf{w} \in V$.)

Prove that any eigenvalue of $V$ is equal to $\pm 1$.

**Problem 8.** Find the general solution $\mathbf{x} \colon \mathbb{R} \to \mathbb{R}^2$ to the inhomogeneous system

$$\dot{\mathbf{x}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t - 1 \\ e^t \end{bmatrix}.$$