MAT 308 Practice Midterm
Spring 2025

Problem 1. Let V be a real vector space, and let $\mathcal{L}(V)$ be the vector space of linear maps $f\colon V \to V$, in which addition and scalar multiplication are defined by $(f+g)(v) = f(v) + g(v)$ and $(r \cdot f)(v) = r \cdot (f(v))$ for $f, g \in \mathcal{L}(V)$, $r \in \mathbb{R}$, and $v \in V$.
Let $m\colon \mathcal{L}(V) \times \mathcal{L}(V) \to \mathcal{L}(V)$ be the map given by composition, so that $m(g, f) = g \circ f$ for $f, g \in \mathcal{L}(V)$.
Prove that m is bilinear.

Problem 2. Find a solution $y\colon \mathbb{R} \to \mathbb{R}$ to the boundary value problem

$$y'' - 4y' + 4y = 0;\ \ y(0) = 0;\ \ y(1) = e^2.$$

Problem 3. Find all solutions $y\colon \mathbb{R} \to \mathbb{R}$ to the initial value problem

$$y' + 2xy = 2x^3; y(1) = 0.$$

Problem 4. Find all solutions $y\colon \mathbb{R} \to \mathbb{R}$ to the differential equation

$$y'' = e^x \sin x$$

Problem 5. Are the following statements true or false? Put a "T" or "F" in the box next to each statement. No justification is necessary.

- [ ] For any linear map $f\colon V \to V$ from a vector space to itself, there is some $\mathbf{v} \in V$ with $f(\mathbf{v}) = \mathbf{v}$.
- [ ] For any function $f\colon \mathbb{R} \to \mathbb{R}$, there exists a differentiable function $F\colon R \to \mathbb{R}$ with $F' = f$.
- [ ] Every finite-dimensional real vector space is isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.
- [ ] The differentiation operator $D\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ on the vector space $\mathcal{C}^\infty(\mathbb{R})$ of smooth functions on $\mathbb{R}$ is injective.
- [ ] If $F(x_0, y_0) > 0$ for all $x_0, y_0 \in \mathbb{R}$, then any solution $y\colon \mathbb{R} \to \mathbb{R}$ to the differential equation $y'(x) = F(x, y(x))$ is injective.
- [ ] If a differential equation $y''(x) = F(x, y(x), y'(x))$ (where $F\colon \mathbb{R}^2 \to \mathbb{R}$) has a solution $y\colon I \to \mathbb{R}$ on some interval $I \subset \mathbb{R}$, then it then it necessarily has a solution $y\colon \mathbb{R} \to \mathbb{R}$ on all of $\mathbb{R}$.
- [ ] Every boundary-value problem of the form $y' = C$; $y(x_0) = y_0$; $y(x_1) = y_1$ where $C, x_0, x_1, y_0, y_1 \in \mathbb{R}$ and $x_0 \neq x_1$ has at least one solution $y\colon \mathbb{R} \to \mathbb{R}$.
- [ ] If $f, g\colon \mathbb{R} \to \mathbb{R}$ are functions such that $f' = g'$ and $f(0) = g(0)$, then $f = g$.
- [ ] Any bilinear map $\beta\colon V \times V \to \mathbb{R}$ from a real vector space V satisfies $\beta(\mathbf{u}, \mathbf{v}) = \beta(\mathbf{v}, \mathbf{u})$ for all $\mathbf{u}, \mathbf{v} \in V$.
- [ ] Every differentiable function is continuous.

Problem 6. Find a solution $y\colon I \to \mathbb{R}$ to the initial value problem

$$y' = y^2 + 1;\ \ y(0) = 0$$

for some interval $I \subset \mathbb{R}$ containing 0.