Notes for MAT 308, Spring 2025

May 7, 13.3: Nonhomogeneous systems

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1 13.3A: Solution formula

- We can handle non-homogeneous linear systems of ODEs just in the same way we handled a single non-homogeneous linear equation, using exponential multipliers.
- First, let us recall the general linear-algebra fact that given any particular solution $\mathbf{x_p}$ to the equation

  $$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}(t),$$

  the *general solution* will be

  $$\mathbf{x} = \mathbf{x}_\mathrm{p} + \mathbf{x}_\mathrm{h},$$

  where $\mathbf{x}_\mathrm{h}$ is a general *homogeneous* solution, i.e., a general solution to

  $$\dot{\mathbf{x}}(t) = A\mathbf{x}(t).$$

- Thus, since we already know how to solve the homogeneous equation, finding the general solution to the inhomogeneous equation amounts to finding any one particular solution.

1.1 Theorem 3.2

- For any $t_0 \in \mathbb{R}$, a particular solution $\mathbf{x}_\mathrm{p}$ to

  $$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}(t)$$

  is given by

  $$\mathbf{x}_\mathrm{p}(t) = e^{tA} \int_{t_0}^{t} e^{-\tau A} \mathbf{b}(\tau)\, \mathrm{d}\tau$$

- Remarks:
  - From the just-mentioned general principle about inhomogeneous equations, it follows that the *general* solution is

    $$\mathbf{x} = \mathbf{x}_\mathrm{p} + \mathbf{x}_\mathrm{h} = e^{tA} \int_{t_0}^{t} e^{-\tau A} \mathbf{b}(\tau)\, \mathrm{d}\tau + e^{tA}\mathbf{c}$$

    with $\mathbf{c} \in K^n$.
  - Note that if we change t_0, this will simply have the effect of modifying the constant $\mathbf{c}$.

– More generally, if we replace $\int_{t_0}^t e^{-\tau A}\mathbf{b}(\tau)\,\mathrm{d}\tau$ with any *antiderivative* of $e^{-tA}\mathbf{b}(t)$, we will obtain a general solution.

  * Unlike in the single-variable case, the *general* antiderivative of a vector-valued function $\mathbf{y}(t)$ is not $\int_{t_0}^t \mathbf{y}(\tau)\,\mathrm{d}\tau$ but rather $\left(\int_{t_1}^t y_1(\tau)\,\mathrm{d}\tau, \dots, \int_{t_n}^t y_n(\tau)\,\mathrm{d}\tau\right)$, i.e., we can choose a different constant of integration on each component.
  * Hence, concretely, instead of $\int_{t_0}^t e^{-\tau A}\mathbf{b}(\tau)\,\mathrm{d}\tau$, we can just choose an antiderivative of each component of $e^{-tA}\mathbf{b}(t)$.
  * Note that in the book, they denote this by $\int e^{-tA}\mathbf{b}(t)\,\mathrm{d}t$, but beware that, as we just said, this does not correspond to a single definite integral, but rather a separate definite integral on each component.
  * Finally, note that if we take a general anti-derivative $\int e^{-tA}\mathbf{b}(t)\,\mathrm{d}t$ in this sense, the result $e^{tA}\int e^{-tA}\mathbf{b}(t)\,\mathrm{d}t$ is actually the *general* solution, since the varying choices of the constant $\mathbf{c}$ in the homogeneous term simply correspond to different choices of anti-derivatives.

- Proof:

  – As mentioned, the proof is just an application of exponential multipliers.

  – We multiply both sides of the equation $\mathbf{x} = A\mathbf{x} + \mathbf{b}$ by the invertible matrix e^{tA} to obtain the equivalent equation

$$e^{-tA}\mathbf{x} - e^{-tA}A\mathbf{x} = e^{tA}\mathbf{b},$$

  which we can also write as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-tA}\mathbf{x}\right) = e^{-tA}\mathbf{b}.$$

  – Applying the fundamental theorem of calculus coordinate-wise, we find a particular solution to this equation by integrating from any t_0:

$$e^{-tA}\mathbf{x}_{\mathrm{p}}(t) = \int_{t_0}^t e^{-\tau A}\mathbf{b}(\tau)\,\mathrm{d}\tau.$$

  – We then solve for $\mathbf{x}_{\mathrm{p}}$:

$$\mathbf{x}_{\mathrm{p}} = e^{tA}\int_{t_0}^t e^{-\tau A}\mathbf{b}(\tau)\,\mathrm{d}\tau.$$

  – As mentioned above, in the penultimate step, we could have instead chosen an arbitrary antiderivative "$\int e^{-tA}\mathbf{b}(t)\,\mathrm{d}t$", and would then arrive instead at the general solution $\mathbf{x}(t) = e^{tA}\int e^{-tA}\mathbf{b}(t)\,\mathrm{d}t$.

1.2 Example 13.3.1

- Let's solve

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}.$$

- We have already considered the corresponding homogeneous equation: since A is in Jordan normal form, we have

$$e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$

  and hence the general homogeneous solution is

$$\mathbf{x}_{\mathrm{h}} = e^{At}\mathbf{c} = \begin{bmatrix} c_1e^t + c_2te^t \\ c_2e^t \end{bmatrix}.$$

- We now compute a particular inhomogeneous solution.
  - We have

$$\begin{aligned}
\mathbf{x}_{\mathrm{p}} &= e^{tA} \int e^{-tA} \mathbf{b}(t) \, \mathrm{d}t \\
&= e^{tA} \int \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix} \mathrm{d}t \\
&= e^{tA} \int \begin{bmatrix} 1 - te^{-2t} \\ e^{-2t} \end{bmatrix} \mathrm{d}t \\
&= e^{tA} \begin{bmatrix} t + \frac{1}{2} te^{-2t} + \frac{1}{4} e^{-2t} \\ -\frac{1}{2} e^{-2t} \end{bmatrix} \\
&= \begin{bmatrix} te^{t} + \frac{1}{4} e^{-t} \\ -\frac{1}{2} e^{-t} \end{bmatrix}
\end{aligned}$$

- Hence, the general solution is

$$\mathbf{x} = \mathbf{x}_{\mathrm{p}} + \mathbf{x}_{\mathrm{h}} = \begin{bmatrix} te^{t} + \frac{1}{4} e^{-t} + c_1 e^{t} + c_2 te^{t} \\ -\frac{1}{2} e^{-t} + c_2 e^{t} \end{bmatrix}.$$