MAT 310 L1 M 8/25

§ Admin stuff: (see syllabus)

Lecturers: Weeks 1-3 Filip Živanović (me)

4-. Joseph Helfer

Office hours: Info here

Course web page:

jojhelfer.com/mat310-2025fall

Recitations: Info here

Weekly homeworks:

- Problems from textbook (free pdf available), see course web page.
- First HW due W Sep 3
- Hand in on Gradescope

Go to gradescope.com

No enroll code needed!

Other important dates:

- Midterm 1: W Oct 1 in class

- Midterm 2: W Nov 12 in class
- Final exam: W Dec 10 Javits 111

After midterm 1, some students will be offered to move up to MAT 315 taught by Prof. Zivanovic (me)

- 315 covers everything in 310 + more.
- Not forced to move up if offered.

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§ Vector spaces

In previous courses, you have solved systems of linear eqns, studied matrices and linear transformations. We will now take a more abstract approach, and discuss <u>vector spaces</u>.

Everyone should feel familiar w the real numbers $\mathbb{R}$ and its properties.
<u>Complex numbers</u> are those of the form $a+bi$, where $i=\sqrt{-1}$. $(i^2=-1)$

The set of complex numbers is denoted by $\mathbb{C}$, and we add/multiply them as follows:

- $(a+bi)+(c+di)=(a+c)+(b+d)i$
- $(a+bi)(c+di)=ac+adi+bci+bdi^2$
  $$=(ac-bd)+(ad+bc)i$$

Both the real and complex numbers satisfy the following properties:

<u>Commutativity</u>:

$$\alpha+\beta=\beta+\alpha \qquad \text{for all } \alpha,\beta\in\mathbb{C}$$
$$\alpha\beta=\beta\alpha$$

<u>associativity</u>:

$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma) \qquad \text{for all}$$
$$(\alpha\beta)\gamma=\alpha(\beta\gamma) \qquad \alpha,\beta,\gamma\in\mathbb{C}$$

<u>identities</u>:

$$\alpha+0=\alpha \qquad \text{for all } \alpha\in\mathbb{C}.$$
$$\alpha\cdot 1=\alpha$$

additive inverse:

For every $\alpha \in \mathbb{C}$, there's a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

(Namely $\beta = -\alpha$!)

multiplicative inverse:

For every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there's a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$

(Namely $\beta = \frac{1}{\alpha} = \frac{1}{a+bi} = \frac{a-bi}{a^2-b^2}$.)

distributive property:

$\gamma(\alpha+\beta) = \gamma\alpha + \gamma\beta$ for all $\alpha, \beta, \gamma \in \mathbb{C}$.

Now, "subtraction" in $\mathbb{C}$ is defined as addition with the additive inverse:

$$\alpha - \beta \text{ is def as } \alpha + (-\beta)$$

Where $-\beta$ is the additive inverse of $\beta \in \mathbb{C}$.

Similarly, "division" is defined as multiplication with the multiplicative inverse: If $\beta \neq 0$

$\frac{\alpha}{\beta}$ is defined as $\alpha \cdot \frac{1}{\beta}$ where

$\frac{1}{\beta} = \beta^{-1}$ is the multiplicative inverse of β.

FROM NOW ON:

$\mathbb{F}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$.

Will call elements of $\mathbb{F}$ "scalars".

Remark: $\mathbb{F}$ is a special case of a mathematical object called "field."

Recall from earlier courses that

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

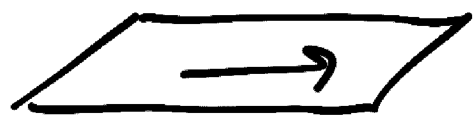
We call elements of $\mathbb{R}^n$ <u>vectors</u>, and sometimes write them as columns

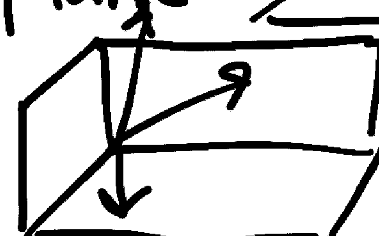
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Similarly we define

$$\mathbb{F}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{F}\}.$$

We visualize $\mathbb{R}$ as a line

$\mathbb{R}^2$ (and $\mathbb{C}$) as a plane 

$\mathbb{R}^3$ as a space 

We may add elements of $\mathbb{F}^n$:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

<u>Theorem.</u> The addition on $\mathbb{F}^n$ is commutative. Meaning

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (y_1, \dots, y_n) + (x_1, \dots, x_n).$$

<u>Proof</u>: For any $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{F}^n$ we have

$(x_1,\dots,x_n)+(y_1,\dots,y_n)$

$$\overset{\text{def}}{=}(x_1+y_1,\dots,x_n+y_n)$$
$$=(y_1+x_1,\dots,y_n+x_n)$$
$$\overset{\text{def}}{=}(y_1,\dots,y_n)+(x_1,\dots,x_n).$$

$\square$

The zero vector in $\mathbb{F}^n$ will often be denoted by

$$\boxed{0=(0,\dots,0)\in\mathbb{F}^n.}$$

As with $\mathbb{F}$, the zero vector $0\in\mathbb{F}^n$ is such that

$$x+0=x \quad \text{for any } x\in\mathbb{F}^n.$$

Vectors have additive inverses:
If $x=(x_1,\dots,x_n)\in\mathbb{F}^n$, then

$-x=(-x_1,\dots,-x_n)$ is the vector such that $\boxed{x+(-x)=0}$

graphically, $-x$ is the vector x w/ direction reversed.

If $\lambda \in \mathbb{F}$ and $x \in \mathbb{F}^n$ we define

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n).$$

It's called Scalar multiplication.

These properties (and some more) make $\mathbb{F}^n$ into an example of a "vector space."

Definition: (Addition, scalar multiplication)
Let V be a set.

(1) An addition on V is a function that assigns an element $a+b \in V$ to each pair of elements $a, b \in V$.

(2) A scalar multiplication on V is a function that assigns an element $\lambda a \in V$ to each $\lambda \in \mathbb{F}$ and $a \in V$.

Definition (Vector space)

A **vector space** is a set V along with an addition on V, and a scalar multiplication, such that the following properties hold:

commutativity:

$a+b=b+a$ for all $a,b \in V$

associativity:

$(a+b)+c=a+(b+c)$ for all $a,b,c \in V$

and

$(\lambda\mu)a=\lambda(\mu a)$ for all $\lambda,\mu \in \mathbb{F}$ and $a \in V$

additive identity:

There's an element $0 \in V$ such that $a+0=a$ for all $a \in V$.

additive inverse:

For every $a \in V$, there's an element $b \in V$ such that $a+b=0$.

multiplicative identity:

$1a = a$ for all $a \in V$

distributive properties:

$\lambda(a+b) = \lambda a + \lambda b$ for all
$(\lambda+\mu)a = \lambda a + \mu a$ $\lambda, \mu \in \mathbb{F}$
$a, b \in V$

Def: We call elements of V "vectors" or "points."

Note that the scalar multiplication on V depends on the scalars.
If $\mathbb{F} = \mathbb{R}$ we say that V is a real vector space (or vector space over $\mathbb{R}$).
If $\mathbb{F} = \mathbb{C}$ we say that V is a complex vector space (or vector space over $\mathbb{C}$).

Ex: • $\{0\}$ is a vector space where $\mathbb{O}=(0,\dots,0)\in\mathbb{F}^n$.
To check it we have to check all the defining properties. Most of them are trivial since

$$\mathbb{O}+0=0.$$

For the multiplicative identity we have

$$1\cdot 0=1(0,\dots,0)=(1\cdot 0,\dots,1\cdot 0)=(0,\dots,0)=0$$

- $\mathbb{R}^n$ is a real vector space
- $\mathbb{C}^n$ is a complex vector space.

Ex. $\mathbb{F}^\infty=\{(x_1,x_2,\dots)\mid x_i\in\mathbb{F}\text{ for all } i\}$ is a vector space. Addition and scalar mult are given by

$$\begin{cases}(x_1,x_2,\dots)+(y_1,y_2,\dots)=(x_1+y_1,x_2+y_2,\dots)\\ \lambda(x_1,x_2,\dots)=(\lambda x_1,\lambda x_2,\dots)\end{cases}$$

One can check that it's a vector sp.

Ex: Let S be a set. Define

$$\mathbb{F}^S = \{f : S \to \mathbb{F} \text{ function}\}$$

Addition & scalar multiplication are defined as follows:

- $f+g \in \mathbb{F}^S$ is the function s.th.

  $(f+g)(x) = f(x) + g(x)$ for all $x \in S$

- $\lambda f \in \mathbb{F}^S$ is the function s.th.

  $(\lambda f)(x) = \lambda f(x)$ for all $x \in S$.

One can check the definition in order to verify that $\mathbb{F}^S$ is a vector space. E.g. the additive identity is the function $0 \in \mathbb{F}^S$ such that $0(x) = 0$ for all $x \in S$.

Proposition: The additive identity in a vector space is unique.

Proof: We need to show that

if 0 and $0'$ are additive identities then $0=0'$.

Therefore assume $0, 0' \in V$ are both additive identities of V. This means

$$\left| \begin{array}{ll} a+0 = a & \text{for all } a \in V \\ b+0' = b & \text{for all } b \in V \end{array} \right.$$

First equality w/ $a=0'$ gives

$$0'+0 = 0' \qquad (1)$$

Second equality w/ $b=0$ gives

$$0+0' = 0 \qquad (2)$$

Addition in V is commutative, so the two left hand sides in (1), (2) are equal. Therefore

$$0' \overset{(1)}{=} 0'+0 \overset{\text{comm.}}{=} 0+0' \overset{(2)}{=} 0.$$

So $0'=0$, and the additive identity must be unique.

$\square$

Prop: Additive inverses in a vector space are unique.

Proof: Assuming $b \in V$ and $b' \in V$ are both additive inverses of $a \in V$, we need to show $b = b'$.

We have

$$\begin{array}{l} a+b+b' = 0+b' = b' \\ \quad \| \\ b+a+b' = b+0 = b \end{array}$$

So $b = b'$, and the additive inverse of a must be unique.

$\square$

Prop: For any $a \in V$ we have
$0 \cdot a = 0$.

Proof: $0 \cdot a = (0+0)a = 0 \cdot a + 0 \cdot a$

Now add the additive inverse to $0 \cdot a$ to both sides:

$$0 = 0 \cdot a$$

$\square$
