

Recall: A vector space (over $\mathbb{F}$) V is a set w/ addition and scalar mult such that

$$(1) u+v = v+u \quad \forall u, v \in V$$

$$(2) (u+v)+w = u+(v+w) \quad \forall u, v, w \in V$$

$$(3) (ab)v = a(bv) \quad \forall a, b \in \mathbb{F}, v \in V$$

$$(4) \text{There's } 0 \in V \text{ such that}$$

$$v+0 = v \quad \forall v \in V$$

$$(5) \text{For every } v \in V \text{ there's } -v \in V \text{ such that } v+(-v)=0$$

$$(6) 1v = v \quad \forall v \in V$$

$$(7) a(v+w) = av+aw \quad \forall a \in \mathbb{F}, v, w \in V$$

$$(8) (a+b)v = av+bv \quad \forall a, b \in \mathbb{F}, v \in V$$

§ Subspaces Let V be a vector space over $\mathbb{F}$.

Def. A subspace of V is a subset $U \subset V$ that is a vector

space itself, with the same addition, scalar mult. and additive identity as in V .

The list of axioms defining a vector space is a little long, but thankfully there's a faster way to check if a subset is a subspace.

Theorem: A subset $U \subset V$ is a subspace if and only if U satisfies the following 3 conditions:

1. (additive identity) $0 \in U$

2. (closed under addition)

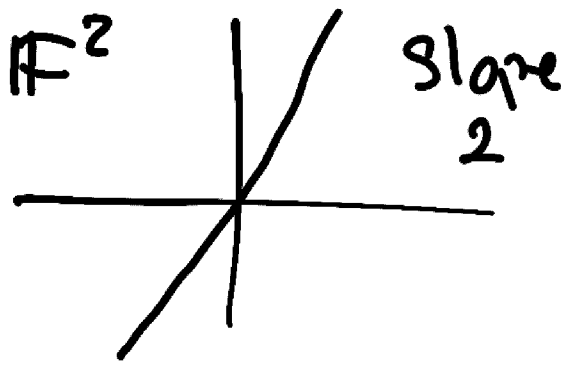
$u, w \in U$ implies $u + w \in U$

3. (closed under scalar mult.)

$a \in \mathbb{F}$ and $u \in U$ implies $au \in U$.

Ex (1) $U = \{(x, y) \in \mathbb{F}^2 \mid y = 2x\}$

$= \{ (x, 2x) \mid x \in \mathbb{F} \} \subset \mathbb{F}^2$
is a subspace.



1. $0 \in U$ by picking
 $x=0$.

2. $(x_1, 2x_1) + (x_2, 2x_2) = (x_1+x_2, 2(x_1+x_2))$
 $\in U$

3. $\lambda(x, 2x) = (\lambda x, 2(\lambda x)) \in U$

So $U \subset \mathbb{F}^2$ is a subspace.

(2) Let $b \in \mathbb{F}$. For what values of b

is $\{ (x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_3 = 5x_4 + b \} \subset \mathbb{F}^4$
a subspace?

1. We need $0 \in U$, and a general
element in U is

$$(x_1, x_2, 5x_4 + b, x_4) \in U$$

Need to pick $x_1 = x_2 = x_4 = 0$, which
gives $(0, 0, b, 0)$, which forces

us to have $\boxed{b=0}$ (else 0 is not in U).

For $b=0$ we can also check that U is closed under addition & scalar mult.

(3) $\{f: [0,1] \rightarrow \mathbb{R} \text{ continuous}\} \subset \mathbb{R}^{[0,1]}$
is a Subspace:

1. $0(x)=0$ is Continuous.
2. If $f, g: [0,1] \rightarrow \mathbb{R}$ are continuous we know from calculus that $f+g: [0,1] \rightarrow \mathbb{R}$ is continuous.
3. Similarly, if $\lambda \in \mathbb{R}$ and $f: [0,1] \rightarrow \mathbb{R}$ is continuous, then so is $\lambda f: [0,1] \rightarrow \mathbb{R}$.

It is very useful to be able to "combine" subspaces. The union of two subspaces is rarely a subspace (see problem in HW #1), so it's not

the kind of operation that is very useful to us. A better operation is the sum of subspaces.

Def (Sum of subspaces)

Let $V_1, \dots, V_m \subset V$ be subspaces of V . The sum of $V_1, \dots, V_m$ is defined as

$$V_1 + \dots + V_m = \{v_1 + \dots + v_m \mid v_1 \in V_1, \dots, v_m \in V_m\}.$$

Sometimes we write it as $\sum_{i=1}^m V_i$.

Ex: $U = \{(x, 0, 0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$ and

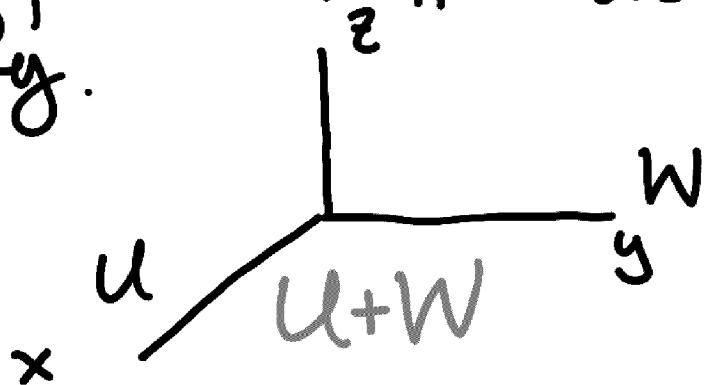
$W = \{(0, y, 0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$ are two subspaces of $\mathbb{F}^3$. Then

$$U + W = \{u + w \mid u \in U, w \in W\}$$

$$= \{(x, 0, 0) + (0, y, 0) \mid (x, 0, 0) \in U, (0, y, 0) \in W\}$$

$$= \{(x, y, 0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$$

Note that $U+W$ is again a subspace of $\mathbb{F}^3$ as you can verify.



Ex: Let $U = \{(x, x, y, y) \in \mathbb{F}^4\} \subset \mathbb{F}^4$

$W = \{(x, x, x, y) \in \mathbb{F}^4\} \subset \mathbb{F}^4.$

and let's find $U+W$. It's the set of all possible sums of elements from U and W . Let

$(x, x, y, y) \in U \quad (z, z, z, w) \in W$

then their sum is

$(x+z, x+z, y+z, y+w) \in U+W.$

only the first two components are equal, so we have

$U+W \subset \{(a,a,b,c) \in \mathbb{F}^4\}$ (subset).

We now show that we in fact have equality. To show it, we must show

$$U+W \supset \{(a,a,b,c) \in \mathbb{F}^4\}. \quad (*)$$

To prove this we must show that any element $(a,a,b,c) \in \mathbb{F}^4$ can be written as a sum

$$(x,x,y,y) + (z,z,z,w) \quad \text{for some } x,y,z,w \in \mathbb{F}.$$

Namely

$$(a,a,b,c) = (a,a,b,b) + (0,0,0,c-b).$$

This shows $(*)$ and hence

$$U+W = \{(x,x,y,z) \in \mathbb{F}^4\}.$$

Can again show $U+W$ is a subspace.

That $U+W$ was a subspace in the two previous examples is not a coincidence.

Theorem: If $V_1, \dots, V_m \subset V$ are subspaces then $V_1 + \dots + V_m \subset V$ is a subspace. In fact, it's the smallest subspace containing $V_1, \dots, V_m$.

Proof: We first check that $\sum_{i=1}^m V_i$ is a subspace:

1. $0 = 0 + \dots + 0 \in \sum_{i=1}^m V_i$

2. If $V_1 + \dots + V_m, W_1 + \dots + W_m \in \sum_{i=1}^m V_i$ then

$$(V_1 + \dots + V_m) + (W_1 + \dots + W_m)$$

$$= (V_1 + W_1) + \dots + (V_m + W_m) \in \sum_{i=1}^m V_i$$

Since $V_i + W_i \in V_i$ for all $i = 1, \dots, m$.

3. If $\lambda \in F$ and $V_1 + \dots + V_m \in \sum_{i=1}^m V_i$

then

$$\lambda(V_1 + \dots + V_m) = (\lambda V_1) + \dots + (\lambda V_m) \in \sum_{i=1}^m V_i$$

since $\lambda V_i \in V_i$ for all $i = 1, \dots, m$.

Therefore $V_1 + \dots + V_m \subset V$ is a subspace. Next we show that it's the Smallest subspace containing $V_1, \dots, V_m$. This means that if

$U \subset V$ is any subspace such that $V_1, \dots, V_m \subset U \subset V$, then $V_1 + \dots + V_m \subset U$.

Since $V_1, \dots, V_m \subset U$, and U is closed under addition, it follows

$V_1 + \dots + V_m \in U$ for any $V_1 \in V_1$
 $\dots, V_m \in V_m$.

Therefore $V_1 + \dots + V_m \subset U$.

□

The sum operation may be further specialized:

Definition (Direct Sum).

Let $V_1, \dots, V_m \subset V$ be subspaces.

The sum $V_1 + \dots + V_m \subset V$ is called a direct sum if each element $w \in \sum_{i=1}^m V_i$ can be written as

$w = v_1 + \dots + v_m$, $v_i \in V_i$ $i = 1, \dots, m$
in only one way. In this case
we use the notation

$$V_1 \oplus \dots \oplus V_m \text{ or } \bigoplus_{i=1}^m V_i$$

Ex: $U = \{(x, 0, 0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$

$$W = \{(0, y, 0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$$

then we saw previously

$U + W = \{(x, y, 0) \in \mathbb{F}^3\}$. This
is in fact a direct sum since

$(x, y, 0) = (x, 0, 0) + (0, y, 0)$ can
not be written as such a sum in

any other way.

$$U \oplus W = \{(x, y, 0) \in \mathbb{F}^3\}.$$

Ex: Let $V_i = \{(0, \dots, 0, \overset{\substack{\uparrow \\ i\text{-th component}}}{x_i}, 0, \dots, 0) \in \mathbb{F}^n\}$
for $i = 1, \dots, m$. Then

$$\mathbb{F}^n = V_1 \oplus \dots \oplus V_m = \bigoplus_{i=1}^m V_i.$$

Ex: $U = \{(x, x) \in \mathbb{F}^2\}$

$$W = \{(x, 0) \in \mathbb{F}^2\}$$

$$Z = \{(x, 2x) \in \mathbb{F}^2\}$$

The sum $U + W + Z$ is not a direct sum. For example the element $(2, 2) \in \mathbb{F}^2$ can be written as

$$(2, 2) = \overset{U}{(2, 2)} + \overset{W}{(0, 0)} + \overset{Z}{(0, 0)}$$

but also as

$$(2, 2) = (0, 0) + (1, 0) + (1, 2)$$

So the sum decomposition of (22) isn't unique.

Proposition: Let $V_1, \dots, V_m \subset V$ be subspaces. The sum $\sum_{i=1}^m V_i$ is a direct sum iff

$$0 = V_1 + \dots + V_m \Rightarrow V_i = 0 \quad i=1, \dots, m.$$

i.e. the only way of writing 0 as a sum is the sum of 0's.

Ex: In our prev example we could've showed $U+W+Z$ isn't a direct sum by writing $(0,0)$ as

$$(0,0) = \underbrace{(2,2)}_U + \underbrace{(-1,0)}_W + \underbrace{(-1,-2)}_Z.$$

Proposition. Let $U, W \subset V$ be subspaces. $U+W$ is a direct sum $\Leftrightarrow U \cap W = \{0\}$

Proof: $\implies$. Assume $U+W$ is a direct sum. Let $v \in U \cap W$. Then we can write 0 as the sum

$$0 = v + (-v)$$

which implies $v = -v = 0$

so $U \cap W = \{0\}$, since our choice of $v \in U \cap W$ was arbitrary.

$\Leftarrow$: Assume $U \cap W = \{0\}$. To

show $U+W$ is a direct sum, we need to show

$$0 = u + w \implies u = w = 0.$$

Namely, the equation $0 = u + w$ implies $w = -u$ (i.e. the additive inverse of u). Since $u \in U$, we have $-u \in U$, so $w = -u \in U$ and thus $w \in U \cap W$, which means $w = 0$, and so $u = 0$. $\square$

Remark/warning: The previous result does not hold for sums of 3 or more vector spaces.
