MAT 310 L1 M 8/25

§Admin stuff: (see syllabus)

Lecturers: Weeks 1-3 Filip Zivanovic (me)

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Office hours: Info here

Course web page:

jojhelfer.com/mat310-2025fall

Recitations: Info here

Weekly homeworks:

- Problems from textbook (free pdf available), see course web page.
- First HW due W Sep 3
- Hand in on Gradescope

Go to gradescope.com

No enroll code needed!

Other important dates:

- Midterm 1: W Oct 1 in class

- Midterm 2: W Nov 12 in class
- Final exam: W Dec 10 Javits 111

After midterm 1, some students will be offered to move up to MAT 315 taught by Prof. Zivanovic (me)

- 315 covers everything in 310 + more.
- Not forced to move up if offered.

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§ Vector spaces

In previous courses, you have solved systems of linear eqns, studied matrices and linear transformations. We will now take a more abstract approach, and discuss vector spaces.

Everyone should feel familiar w the real numbers $\mathbb{R}$ and its properties. Complex numbers are those of the form $a+bi$, where $i=\sqrt{-1}$. ($i^2=-1$)

The set of complex numbers is denoted by $\mathbb{C}$, and we add/multiply them as follows:

- $(a+bi)+(c+di)=(a+c)+(b+d)i$;
- $(a+bi)(c+di)=ac+adi+bci+bdi^2$
$$=(ac-bd)+(ad+bc)i$$

Both the real and complex numbers satisfy the following properties:

<u>Commutativity</u>:

$$\alpha+\beta=\beta+\alpha \qquad \text{for all } \alpha,\beta\in\mathbb{C}$$
$$\alpha\beta=\beta\alpha$$

<u>associativity</u>:

$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma) \qquad \text{for all}$$
$$(\alpha\beta)\gamma=\alpha(\beta\gamma) \qquad \alpha,\beta,\gamma\in\mathbb{C}$$

<u>identities</u>:

$$\alpha+0=\alpha \qquad \text{for all } \alpha\in\mathbb{C}.$$
$$\alpha\cdot 1=\alpha$$

__additive inverse:__

For every $\alpha \in \mathbb{C}$, there's a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

(Namely $\beta = -\alpha$!)

__multiplicative inverse:__

For every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there's a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$

(Namely $\beta = \frac{1}{\alpha} = \frac{1}{a+bi} = \frac{a-bi}{a^2-b^2}$.)

__distributive property:__

$\gamma(\alpha+\beta) = \gamma\alpha + \gamma\beta$ for all $\alpha, \beta, \gamma \in \mathbb{C}$.

Now, "__subtraction__" in $\mathbb{C}$ is defined as addition with the additive inverse:

$$\alpha - \beta \text{ is def as } \alpha + (-\beta)$$

Where $-\beta$ is the additive inverse of $\beta \in \mathbb{C}$.

Similarly, "division" is defined as multiplication with the multiplicative inverse: If $\beta \neq 0$

$\frac{\alpha}{\beta}$ is defined as $\alpha \cdot \frac{1}{\beta}$ where

$\frac{1}{\beta} = \beta^{-1}$ is the multiplicative inverse of β.

FROM NOW ON:

$\mathbb{F}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$.

Will call elements of $\mathbb{F}$ "scalars".

Remark: $\mathbb{F}$ is a special case of a mathematical object called "field."

Recall from earlier courses that

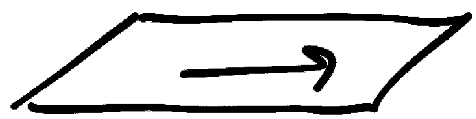
$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

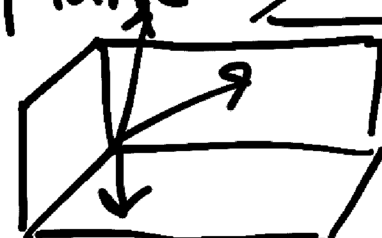
We call elements of $\mathbb{R}^n$ <u>vectors</u>, and sometimes write them as columns

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$ Similarly we define

$$\mathbb{F}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{F}\}.$$

We visualize $\mathbb{R}$ as a line

$\mathbb{R}^2$ (and $\mathbb{C}$) as a plane 

$\mathbb{R}^3$ as a space 

We may add elements of $\mathbb{F}^n$:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

<u>Theorem.</u> The addition on $\mathbb{F}^n$ is commutative. Meaning

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (y_1, \dots, y_n) + (x_1, \dots, x_n).$$

<u>Proof</u>: For any $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{F}^n$ we have

$(x_1,\dots,x_n)+(y_1,\dots,y_n)$

$$\overset{\text{def}}{=}(x_1+y_1,\dots,x_n+y_n)$$
$$=(y_1+x_1,\dots,y_n+x_n)$$
$$\overset{\text{def}}{=}(y_1,\dots,y_n)+(x_1,\dots,x_n).$$

$\square$

The zero vector in $\mathbb{F}^n$ will often be denoted by

$$\boxed{0=(0,\dots,0)\in\mathbb{F}^n.}$$

As with $\mathbb{F}$, the zero vector $0\in\mathbb{F}^n$ is such that

$$x+0=x \quad \text{for any } x\in\mathbb{F}^n.$$

Vectors have additive inverses:

If $x=(x_1,\dots,x_n)\in\mathbb{F}^n$, then

$-x=(-x_1,\dots,-x_n)$ is the vector such that $\boxed{x+(-x)=0}$

graphically, $-x$ is the vector x w/ direction reversed.

If $\lambda \in \mathbb{F}$ and $x \in \mathbb{F}^n$ we define

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n).$$

It's called Scalar multiplication.

These properties (and some more) make $\mathbb{F}^n$ into an example of a "vector space."

<u>Definition</u>: (Addition, scalar multiplication)
Let V be a set.

(1) An <u>addition</u> on V is a function that assigns an element $a+b \in V$ to each pair of elements $a, b \in V$.

(2) A <u>scalar multiplication</u> on V is a function that assigns an element $\lambda a \in V$ to each $\lambda \in \mathbb{F}$ and $a \in V$.

Definition (Vector space)

A vector space is a set V along with an addition on V, and a scalar multiplication, such that the following properties hold:

commutativity:

$$a+b=b+a \text{ for all } a,b\in V$$

associativity:

$$(a+b)+c=a+(b+c) \quad \text{for all } a,b,c\in V$$

and

$$(\lambda\mu)a=\lambda(\mu a) \quad \text{for all } \lambda,\mu\in\mathbb{F} \text{ and } a\in V$$

additive identity:

There's an element $0\in V$ such that $a+0=a$ for all $a\in V$.

additive inverse:

For every $a\in V$, there's an element $b\in V$ such that $a+b=0$.

multiplicative identity:

$$1a = a \quad \text{for all } a \in V$$

distributive properties:

$$\lambda(a+b) = \lambda a + \lambda b$$
$$(\lambda+\mu)a = \lambda a + \mu a$$

for all $\lambda, \mu \in \mathbb{F}$, $a, b \in V$

Def: We call elements of V "vectors" or "points."

Note that the scalar multiplication on V depends on the scalars.

If $\mathbb{F} = \mathbb{R}$ we say that V is a real vector space (or vector space over $\mathbb{R}$).

If $\mathbb{F} = \mathbb{C}$ we say that V is a complex vector space (or vector space over $\mathbb{C}$).

Ex: • $\{0\}$ is a vector space where $0=(0,\dots,0)\in\mathbb{F}^n$.
To check it we have to check all the defining properties. Most of them are trivial since

$$0+0=0.$$

For the multiplicative identity we have

$$1\cdot 0=1(0,\dots,0)=(1\cdot 0,\dots,1\cdot 0)=(0,\dots,0)=0$$

• $\mathbb{R}^n$ is a real vector space
• $\mathbb{C}^n$ is a complex vector space.

Ex. $\mathbb{F}^\infty=\{(x_1,x_2,\dots)\mid x_i\in\mathbb{F} \text{ for all } i\}$ is a vector space. Addition and scalar mult are given by

$$(x_1,x_2,\dots)+(y_1,y_2,\dots)=(x_1+y_1,x_2+y_2,\dots)$$
$$\lambda(x_1,x_2,\dots)=(\lambda x_1,\lambda x_2,\dots)$$

One can check that it's a vector sp.

Ex: Let S be a set. Define

$$\mathbb{F}^S = \{f : S \to \mathbb{F} \text{ function}\}$$

Addition & scalar multiplication are defined as follows:

- $f+g \in \mathbb{F}^S$ is the function s.th.

  $(f+g)(x) = f(x) + g(x)$ for all $x \in S$

- $\lambda f \in \mathbb{F}^S$ is the function s.th.

  $(\lambda f)(x) = \lambda f(x)$ for all $x \in S$.

One can check the definition in order to verify that $\mathbb{F}^S$ is a vector space. E.g. the additive identity is the function $0 \in \mathbb{F}^S$ such that $0(x) = 0$ for all $x \in S$.

Proposition: The additive identity in a vector space is unique.

Proof: We need to show that

if 0 and $0'$ are additive identities then $0=0'$.

Therefore assume $0, 0' \in V$ are both additive identities of V. This means

$$\left| \begin{array}{ll} a+0=a & \text{for all } a \in V \\ b+0'=b & \text{for all } b \in V \end{array} \right.$$

First equality w/ $a=0'$ gives

$$0'+0=0' \qquad (1)$$

Second equality w/ $b=0$ gives

$$0+0'=0 \qquad (2)$$

Addition in V is commutative, so the two left hand sides in (1), (2) are equal. Therefore

$$0' \overset{(1)}{=} 0'+0 \overset{\text{comm.}}{=} 0+0' \overset{(2)}{=} 0.$$

So $0'=0$, and the additive identity must be unique. $\square$

Prop: Additive inverses in a vector space are unique.

Proof: Assuming $b \in V$ and $b' \in V$ are both additive inverses of $a \in V$, we need to show $b = b'$.

We have

$$\begin{array}{c} a+b+b' = 0+b' = b' \\ \| \\ b+a+b' = b+0 = b \end{array}$$

So $b = b'$, and the additive inverse of a must be unique.

$\square$

Prop: For any $a \in V$ we have $0 \cdot a = 0$.

Proof: $0 \cdot a = (0+0)a = 0 \cdot a + 0 \cdot a$

Now add the additive inverse to $0 \cdot a$ to both sides:

$$0 = 0 \cdot a$$

$\square$

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<u>Recall</u>: A vector space (over $\mathbb{F}$) V is a set w/ addition and scalar mult such that

(1) $u+v = v+u \quad \forall u,v \in V$

(2) $(u+v)+w = u+(v+w) \quad \forall u,v,w \in V$

(3) $(ab)v = a(bv) \quad \forall a,b \in \mathbb{F},\ v \in V$

(4) There's $0 \in V$ such that
$$v+0 = v \quad \forall v \in V$$

(5) For every $v \in V$ there's $-v \in V$ such that $v+(-v) = 0$

(6) $1v = v \quad \forall v \in V$

(7) $a(v+w) = av+aw \quad \forall a \in \mathbb{F},\ v,w \in V$

(8) $(a+b)v = av+bv \quad \forall a,b \in \mathbb{F},\ v \in V$

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§ <u>Subspaces</u> Let V be a vector space over $\mathbb{F}$.

<u>Def</u>: A <u>subspace</u> of V is a subset $U \subset V$ that is a vector

space itself, with the same addition, scalar mult. and additive identity as in V.

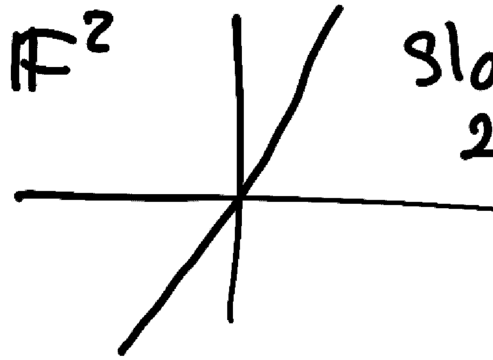
The list of axioms defining a vector space is a little long, but thankfully there's a faster way to check if a subset is a subspace.

Theorem: A subset $U \subset V$ is a subspace if and only if U satisfies the following 3 conditions:

1. (additive identity) $0 \in U$
2. (closed under addition) $u, w \in U$ implies $u + w \in U$
3. (closed under scalar mult.) $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$.

Ex (1) $U = \{(x, y) \in \mathbb{F}^2 \mid y = 2x\}$

$= \{(x, 2x) \mid x \in \mathbb{F}\} \subset \mathbb{F}^2$
is a subspace.



1. $0 \in U$ by picking $x = 0$.

2. $(x_1, 2x_1) + (x_2, 2x_2) = (x_1 + x_2, 2(x_1 + x_2)) \in U$

3. $\lambda(x, 2x) = (\lambda x, 2(\lambda x)) \in U$

so $U \subset \mathbb{F}^2$ is a subspace.

(2) Let $b \in \mathbb{F}$. For what values of b is $\underbrace{\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_3 = 5x_4 + b\}}_{U} \subset \mathbb{F}^4$ a subspace?

1. We need $0 \in U$, and a general element in U is

$$(x_1, x_2, 5x_4 + b, x_4) \in U$$

Need to pick $x_1 = x_2 = x_4 = 0$, which gives $(0, 0, b, 0)$, which forces us to have $\boxed{b=0}$ (else 0 is not in U).

For $b=0$ we can also check that U is closed under addition & scalar mult.

(3) $\{f:[0,1]\to\mathbb{R} \text{ continuous}\}\subset\mathbb{R}^{[0,1]}$ is a subspace:

1. $0(x)=0$ is continuous.
2. If $f,g:[0,1]\to\mathbb{R}$ are continuous we know from calculus that $f+g:[0,1]\to\mathbb{R}$ is continuous
3. Similarly, if $\lambda\in\mathbb{R}$ and $f:[0,1]\to\mathbb{R}$ is continuous, then so is $\lambda f:[0,1]\to\mathbb{R}$.

It is very useful to be able to "combine" subspaces. The union of two subspaces is rarely a subspace (see problem in HW#1), so it's not the kind of operation that is very useful to us. A better operation is the <u>sum</u> of subspaces.

<u>Def</u> (Sum of subspaces)
Let $V_1, \ldots, V_m \subset V$ be subspaces of V. The <u>sum</u> of $V_1, \ldots, V_m$ is defined as

$$V_1 + \cdots + V_m = \{ v_1 + \cdots + v_m \mid v_1 \in V_1, \ldots, v_m \in V_m \}.$$

Sometimes we write it as $\sum_{i=1}^{m} V_i$.

<u>Ex</u>: $U = \{ (x,0,0) \in \mathbb{F}^3 \} \subset \mathbb{F}^3$ and

$W = \{ (0,y,0) \in \mathbb{F}^3 \} \subset \mathbb{F}^3$ are

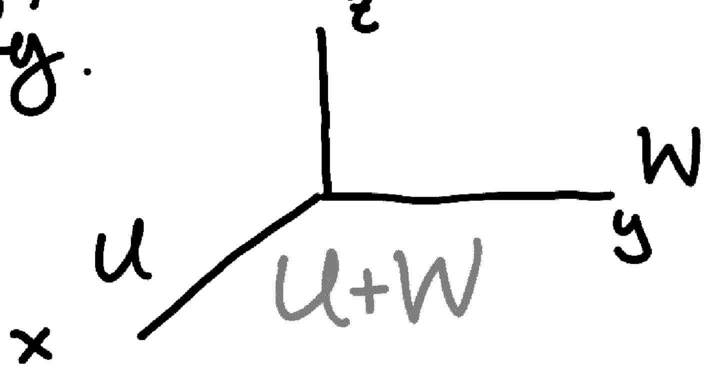
two subspaces of $\mathbb{F}^3$. Then

$$U + W = \{ u + w \mid u \in U, w \in W \}$$

$$= \{ (x,0,0) + (0,y,0) \mid (x,0,0) \in U, (0,y,0) \in W \}$$

$$= \{ (x,y,0) \in \mathbb{F}^3 \} \subset \mathbb{F}^3$$

Note that $U+W$ is again a subspace of $\mathbb{F}^3$ as you can verify.



Ex: Let $U=\{(x,x,y,y)\in\mathbb{F}^4\}\subset\mathbb{F}^4$

$W=\{(x,x,x,y)\in\mathbb{F}^4\}\subset\mathbb{F}^4$.

and let's find $U+W$. It's the set of all possible sums of elements from U and W. Let

$$(x,x,y,y)\in U \quad (z,z,z,w)\in W$$

then their sum is

$$(x+z,x+z,y+z,y+w)\in U+W.$$

Only the first two components are equal, so we have

$U+W \subset \{(a,a,b,c) \in \mathbb{F}^4\}$ (subset).
We now show that we in fact have equality. To show it, we must show

$$U+W \supset \{(a,a,b,c) \in \mathbb{F}^4\}. \quad (*)$$

To prove this we must show that any element $(a,a,b,c) \in \mathbb{F}^4$ can be written as a sum

$$(x,x,y,y) + (z,z,z,w) \text{ for some } x,y,z,w \in \mathbb{F}.$$

Namely

$$(a,a,b,c) = (a,a,b,b) + (0,0,0,c-b).$$

This shows $(*)$ and hence

$$U+W = \{(x,x,y,z) \in \mathbb{F}^4\}.$$

Can again show $U+W$ is a subspace.

That $U+W$ was a subspace in the two previous examples is not a coincidence.

<u>Theorem</u>: If $V_1, \dots, V_m \subset V$ are subspaces then $V_1 + \dots + V_m \subset V$ is a subspace. In fact, it's the <u>smallest</u> subspace containing $V_1, \dots, V_m$.

<u>Proof</u>: We first check that $\sum_{i=1}^{m} V_i$ is a subspace:

1. $0 = 0 + \dots + 0 \in \sum_{i=1}^{m} V_i$
2. If $v_1 + \dots + v_m,\ w_1 + \dots + w_m \in \sum_{i=1}^{m} V_i$ then

$$(v_1 + \dots + v_m) + (w_1 + \dots + w_m)$$

$$= (v_1 + w_1) + \dots + (v_m + w_m) \in \sum_{i=1}^{m} V_i$$

since $v_i + w_i \in V_i$ for all $i = 1, \dots, m$.

3. If $\lambda \in \mathbb{F}$ and $v_1 + \dots + v_m \in \sum_{i=1}^{m} V_i$

then

$$\lambda(v_1+\cdots+v_m)=(\lambda v_1)+\cdots+(\lambda v_m)\in\sum_{i=1}^{m}V_i$$

since $\lambda v_i \in V_i$ for all $i=1,\dots,m$.

Therefore $V_1+\cdots+V_m \subset V$ is a subspace. Next we show that it's the smallest subspace containing $V_1,\dots,V_m$. This mean that if

$U \subset V$ is any subspace

such that $V_1,\dots,V_m \subset U \subset V$, then $V_1+\cdots+V_m \subset U$.

Since $V_1,\dots,V_m \subset U$, and U is closed under addition, it follows $v_1+\cdots+v_m \in U$ for any $v_1 \in V_1$ $,\dots, v_m \in V_m$.

Therefore $V_1+\cdots+V_m \subset U$.

$\square$

The sum operation may be further specialized:

Definition (Direct Sum).

Let $V_1, \dots, V_m \subset V$ be subspaces.

The sum $V_1 + \cdots + V_m \subset V$ is called a direct sum if each element $w \in \sum_{i=1}^{m} V_i$ can be written as

$$w = v_1 + \cdots + v_m \ , \ v_i \in V_i \qquad i = 1, \dots, m$$

in only one way. In this case we use the notation

$$V_1 \oplus \cdots \oplus V_m \text{ or } \bigoplus_{i=1}^{m} V_i$$

Ex: $U = \{(x,0,0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$

$W = \{(0,y,0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$

then we saw previously $U + W = \{(x,y,0) \in \mathbb{F}^3\}$. This is in fact a direct sum since

$(x,y,0) = (x,0,0) + (0,y,0)$ can not be written as such a sum in any other way.

$$U \oplus W = \{(x,y,0) \in \mathbb{F}^3\}.$$

<u>Ex</u>: Let $V_i = \{(0,\dots,0,\underset{\substack{\uparrow \\ i\text{-th component}}}{x_i},0,\dots,0) \in \mathbb{F}^n\}$

for $i = 1,\dots,m$. Then

$$\mathbb{F}^n = V_1 \oplus \cdots \oplus V_m = \bigoplus_{i=1}^{m} V_i.$$

<u>Ex</u>: $U = \{(x,x) \in \mathbb{F}^2\}$

$W = \{(x,0) \in \mathbb{F}^2\}$

$Z = \{(x,2x) \in \mathbb{F}^2\}$

The sum $U+W+Z$ is <u>not</u> a direct sum. For example the element $(2,2) \in \mathbb{F}^2$ can be written as

$$(2,2) = \overset{U}{(2,2)} + \overset{W}{(0,0)} + \overset{Z}{(0,0)}$$

but also as

$$(2,2) = (0,0) + (1,0) + (1,2)$$

So the sum decomposition of $(2,2)$ isn't unique.

<u>Proposition</u>: Let $V_1, \dots, V_m \subset V$ be subspaces. The sum $\sum_{i=1}^{m} V_i$ is a direct sum iff

$$0 = v_1 + \dots + v_m \Rightarrow v_i = 0 \quad i = 1, \dots, m.$$

i.e. the only way of writing 0 as a sum is the sum of 0's.

<u>Ex</u>: In our prev example we could've showed $U+W+Z$ isn't a direct sum by writing $(0,0)$ as

$$(0,0) = \underbrace{(2,2)}_{U} + \underbrace{(-1,0)}_{W} + \underbrace{(-1,-2)}_{Z}.$$

<u>Proposition</u>. Let $U, W \subset V$ be subspaces. $U+W$ is a direct sum $\Leftrightarrow U \cap W = \{0\}$

Proof: $\Longrightarrow$. Assume $\mathcal{U}+W$ is a direct sum. Let $v \in \mathcal{U} \cap W$. Then we can write 0 as the sum

$$0 = v + (-v)$$

which implies $v = -v = 0$

so $\mathcal{U} \cap W = \{0\}$, since our choice of $v \in \mathcal{U} \cap W$ was arbitrory.

$\Longleftarrow$: Assume $\mathcal{U} \cap W = \{0\}$. To show $\mathcal{U}+W$ is a direct sum, we need to show

$$0 = u + w \implies u = w = 0.$$

Namely, the equation $0 = u + w$ implies $w = -u$ (i.e. the additive inverse of u). Since $u \in \mathcal{U}$, we have $-u \in \mathcal{U}$, so $w = -u \in \mathcal{U}$ and thus $w \in \mathcal{U} \cap W$, which means $w = 0$, and so $u = 0$. $\square$

<u>Remark/warning</u>: The previous result does <u>not</u> hold for sums of 3 or more vector spaces.

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§Span and linear independence

Let V be a vector field over $\mathbb{F}$.

<u>Def</u>: A linear combination of vectors $v_1, \dots, v_m$ is a vector of the form

$$a_1v_1 + \dots + a_mv_m = \sum_{i=1}^{m} a_iv_i \quad \begin{matrix} a_i \in \mathbb{F} \\ v_i \in V \end{matrix}$$

<u>Ex</u>: • $(17,-4,2)$ is a lin. comb. of $(2,1,-3)$ and $(1,-2,4)$ because

$$(17,-4,2) = 6(2,1,-2) + 5(1,-2,4).$$

• $(17,-4,5)$ is <u>not</u> a lin comb of $(2,1,-3)$ and $(1,-2,4)$ because the equation

$$(17,-4,5) = a_1(2,1,-3) + a_2(1,-2,4)$$

has no solutions. It's equivalent to the system of linear eqns

$$\begin{cases} 17 = 2a_1 + a_2 \\ -4 = a_1 - 2a_2 \\ 5 = -3a_1 + 4a_2 \end{cases}$$ which has no sol as you can check.

<u>Def</u>: The <u>span</u> of $v_1, \dots, v_m \in V$ is the set of all linear combinations

$$\operatorname{span}(v_1, \dots, v_m) = \left\{ \sum_{i=1}^{m} a_i v_i \;\middle|\; a_i \in \mathbb{F} \ \forall i \right\}.$$

<u>Ex</u> From previous ex

$v_1 = (2, 1, -3)$, $v_2 = (1, -2, 4)$,

$w_1 = (17, -4, 2)$, $w_2 = (17, -4, 5)$

and we have $w_1 \in \operatorname{span}(v_1, v_2)$ but $w_2 \notin \operatorname{span}(v_1, v_2)$.

<u>Def</u>: A vector space V is <u>finite dimensional</u> if V is the span of a finite list of vectors, i.e.,

$V = \operatorname{span}(v_1, \dots, v_m)$ for some $v_1, \dots, v_m \in V$.

Def: If V is not finite dim we say that it's infinite dim.

Ex: • $\mathbb{F}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{F} \ \forall i\}$ is finite dim since

$\mathbb{F}^n = \text{span}\big((1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,\dots,0,1)\big)$

• $\mathbb{F}^\infty = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{F} \ \forall i\}$ is infinite dimensional.

Def: • Let $\mathcal{P}(\mathbb{F})$ denote the set of all polynomials w/ coefficients in $\mathbb{F}$.

• Let $\mathcal{P}_m(\mathbb{F})$ be the set of all polynomials of degree $\leq m$.

Prop. $\mathcal{P}(\mathbb{F})$ is infinitely dimensional.

Proof: Going for a contradiction, assume it's not & write

$$\mathcal{P}(\mathbb{F}) = \text{span}(p_1, \dots, p_m).$$

Let $D = \max_{1 \le i \le m} \deg p_i$. This exists because each (non-zero) polynomial has a non-negative degree. But then $x^{D+1} \notin \operatorname{span}(p_1, \dots, p_m)$ which yields a contradiction. $\square$

<u>Def</u> A set of vectors $v_1, \dots, v_m$ is said to be <u>linearly independent</u> iff

$$\sum_{i=1}^{m} a_i v_i = 0 \Rightarrow a_i = 0 \ \forall i.$$

If they are not linearly independent, they are called linearly dependent.

<u>Rmk</u>: A set of vectors $v_1, \dots, v_m$ is linearly independent if $w \in \operatorname{span}(v_1, \dots, v_m)$ can only be written as a linear combination in <u>one</u> way.

(Compare this to sum/direct sum of subspaces.)

Ex: (a) $(1,0,0)$ and $(0,1,0)$ are linearly independent:

$$a_1(1,0,0)+a_2(0,1,0)=0$$

$$\Leftrightarrow \quad (a_1,a_2,0) = (0,0,0)$$

$$\Leftrightarrow \quad a_1=a_2=0.$$

(b) Let m be a non-negative integer. Then $1, z, \dots, z^m$ are linearly independent in $\mathcal{P}(\mathbb{F})$. Then

$$a_0+a_1z+\cdots+a_mz^m=0 \quad \text{means}$$

an equality in $\mathcal{P}(\mathbb{F})$, so this equality should be true for all $z \in \mathbb{F}$. The only option is

$$a_0=a_1=\cdots=a_m=0.$$

(c) Any non-zero $v \in V$ is linearly independent since $av=0 \Rightarrow a=0$.

(d) The zero vector $0 \in V$ is linearly dependent since eg $1 \cdot 0 = 0$.

(e) V_1 and V_2 are linearly dependent iff $V_1 = \lambda V_2$ for some $\lambda \in \mathbb{F}$.

(f) $(2,3,1)$, $(1,-1,2)$, $(7,3,8)$ is lin dep in $\mathbb{F}^3$ since

$2(2,3,1) + 3(1,-1,2) + (-1)(7,3,8) = 0$

i.e. we can write 0 as a non-trivial linear combination.

<u>Prop</u>: If $V_1, \dots, V_m$ is lin dep. Then $\exists k \in \{1, \dots, m\}$ s.th.

$$V_k \in \operatorname{span}(V_1, \dots, V_{k-1}).$$

Furthermore if V_k is such a vector then $\operatorname{span}(V_1, \dots, V_m) = \operatorname{span}(V_1, \dots, \hat{V}_k, \dots, V_m)$

($v_1, \dots, \hat{v}_k, \dots, v_m$ is the list $v_1, \dots, v_m$ with v_k removed.)

<u>Proof</u>: By assumption $v_1, \dots, v_m$ is lin dep so $\exists a_1, \dots, a_m$ not all $= 0$ such that $a_1 v_1 + \dots + a_m v_m = 0$. Let $k = \max\{i \mid a_i \neq 0\}$. Then

$$a_1 v_1 + \dots + a_m v_m = a_1 v_1 + \dots + a_k v_k = 0$$

with $a_k \neq 0$. So:

$$v_k = -\frac{a_1}{a_k} v_1 - \dots - \frac{a_{k-1}}{a_k} v_{k-1}$$

$\Leftrightarrow$ $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$. For the second part, it's obvious that

$$\operatorname{span}(v_1, \dots, \hat{v}_k, \dots, v_m) \subset \operatorname{span}(v_1, \dots, v_m)$$

so it suffices to prove

$$\operatorname{span}(v_1, \dots, \hat{v}_k, \dots, v_m) \supset \operatorname{span}(v_1, \dots, v_m)$$

Let $v = a_1 v_1 + \dots + a_m v_m \in \operatorname{span}(v_1, \dots, v_m)$.

Since we now assume $v_k \in \text{span}(v_1, \ldots, v_{k-1})$ so $v_k = b_1 v_1 + \cdots + b_{k-1} v_{k-1}$, and

$$v = a_1 v_1 + \cdots + a_k v_k + \cdots + a_m v_m$$

$$= a_1 v_1 + \cdots + a_k (b_1 v_1 + \cdots + b_{k-1} v_{k-1}) + \cdots + a_m v_m$$

$$= (a_1 + a_k b_1) v_1 + \cdots + (a_{k-1} + a_k b_{k-1}) v_{k-1}$$

$$+ a_{k+1} v_{k+1} + \cdots + a_m v_m$$

$\in \text{Span}(v_1, \ldots, \hat{v}_k, \ldots, v_m)$ since this is now a lin comb not involving v_k anymore. This shows

$$\text{span}(v_1, \ldots, \hat{v}_k, \ldots, v_m) \supset \text{Span}(v_1, \ldots, v_m)$$

and hence

$$\text{span}(v_1, \ldots, \hat{v}_k, \ldots, v_m) = \text{Span}(v_1, \ldots, v_m)$$

$\square$

<u>Ex</u>: Consider $v_1 = (1,2,3)$, $v_2 = (6,5,4)$, $v_3 = (15,16,17)$, $v_4 = (8,9,7)$. Let's find the smallest k s.t.h. $v_k \in \text{span}\{v_1, \ldots, v_{k-1}\}$.

$\underline{k=1}$: $v_1 \neq 0$ so v_1 is lin indep.

$\underline{k=2}$: We see that v_2 is not a scalar multiple of v_1:

$$(6,5,4) = c(1,2,3) \text{ means } \begin{cases} c = 6 \\ 2c = 5 \\ 3c = 4 \end{cases}$$

which obviously has no sol.

$\underline{k=3}$: Now, we <u>do</u> have

$v_3 = av_1 + bv_2$ for $a=3$, $b=2$, so

$v_3 \in \operatorname{span}(v_1, v_2)$ which means that v_1, v_2, v_3, v_4 is lin dep &

$$\operatorname{Span}(v_1, v_2, v_3, v_4) = \operatorname{Span}(v_1, v_2, v_4)$$

by the previous proposition.

the "linear dependence lemma"

$\underline{\text{Prop}}$: In a fin. dim Vector space, the length of every lin independent list is less than or equal to the length of every spanning list.

Proof: Supp. $u_1, \dots, u_m$ is lin indep and $w_1, \dots, w_n$ spans V. We will show $m \leq n$.

Step 1: Because $u_1 = a_1 w_1 + \dots + a_n w_n$ for some $a_j \in \mathbb{F}$, the list

$u_1, w_1, \dots, w_n$ is lin dep. Since $u_1 \neq 0$, the lin dep lemma implies that there is some w_k such that $w_k \in \operatorname{Span}(u_1, w_1, \dots, w_{k-1})$ and

$$\operatorname{Span}(u_1, w_1, \dots, w_n) = \operatorname{Span}(u_1, w_1, \dots, \hat{w}_k, \dots, w_n)$$
$$= V$$

Repeat this process. At step k $(k \geq 2)$ the list from step $k-1$ is

$u_1, \dots, u_{k-1}, w_{i_k}, \dots, w_{i_n}$ (n vectors in total)

& its span $= V$. So

$$u_k = a_1 u_1 + \dots + a_{k-1} u_{k-1} + a_k w_{i_k} + \dots + a_n w_{i_n}$$

So the new list

$u_1, \ldots, u_{k-1}, u_k, w_{j_k}, \ldots, w_{j_n}$ is lin dep.

becuse $u_1, \ldots, u_k$ is lin indep, $\exists$ some $w_j \in \operatorname{span}(u_1, \ldots, u_k, w_{j_k}, \ldots, w_{j-1})$

Now at the final step when we have the list

$u_1, \ldots, u_m, w_{j_m}, \ldots, w_{j_n}$ it must

Still be lin dep by the above reasoning. But since $(u_1, \ldots, u_m)$ is lin indep, the list of w's <u>can not</u> exhausted yet, meaning $m \geq n$.

$\square$

<u>Ex</u>: • Since $(1,0)$ and $(0,1)$ spans $\mathbb{F}^2$, no list of ≥ 3 vectors in $\mathbb{F}^2$ can be linearly dependent.

e.g. $(1,2), (-1,5), (7,0)$ must be lin dep.

- Since $(1,0,0),(0,1,0),(0,0,1)$ are linearly indep in $\mathbb{F}^3$, no list of ≤ 2 vectors in $\mathbb{F}^3$ can span it!

<u>Prop</u>: Every subspace of a fin dim vector space is finite dimensional

Recall: • $v_1, \dots, v_n$ lin indep iff

$$a_1 v_1 + \cdots + a_n v_n = 0 \Rightarrow a_i = 0 \ \forall i$$

• $V = \operatorname{span}(v_1, \dots, v_n)$ iff $\forall u \in V$

$\exists b_1, \dots, b_n \in \mathbb{F}$: $u = b_1 v_1 + \cdots + b_n v_n$.

§ Bases

Def: A basis of V is a linearly independent spanning set of vectors

Ex: (a) $(1,0), (0,1)$ is a basis for $\mathbb{F}^2$

(b) $(1,0,\dots,0), \dots, (0,\dots,0,1)$ is a basis for $\mathbb{F}^n$. ("standard basis")

(c) $(1,2), (1,0)$ is a basis for $\mathbb{F}^2$.

(d) $1, z, \dots, z^m$ is a basis (the "standard basis") for $P_m(\mathbb{F})$.

Prop: A list $v_1,\dots,v_n$ is a basis for V iff $\forall u \in V$ can be written uniquely on the form

$$u = a_1 v_1 + \cdots + a_n v_n.$$

Proof: $\underline{\Longrightarrow}$: Assume $v_1,\dots,v_n$ is a basis. Since it spans every $u \in V$ can be written as

$$u = a_1 v_1 + \cdots + a_n v_n.$$

If it could also be written as

$$u = b_1 v_1 + \cdots + b_n v_n$$

we have

$$0 = u - u = (a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n$$

$\Rightarrow a_i = b_i \ \forall i$ because of lin indep

$\underline{\Longleftarrow}$: Since every $u \in V$ has a unique representation

$u = a_1 v_1 + \cdots + a_n v_n$ it means $v_1,\dots,v_n$ spans.

In particular we always have
$0 = 0 \cdot v_1 + \cdots + 0 \cdot v_n$ & becuse this is the unique way of representing 0, $v_1, \ldots, v_n$ must also be lin indep. $\square$

<u>Prop</u>: Every spanning list can be reduced to a basis for V.

<u>Proof</u>: Let $V = \mathrm{span}(v_1, \ldots, v_n)$, and call this spanning list B. Define

$$B' := \{ v_k \mid v_k \notin \mathrm{span}(v_1, \ldots, v_{k-1}) \}$$

(Note $\mathrm{span}() = \{0\}$ by def.)

Then by the linear dep lemma from last time $\mathrm{span}\, B = \mathrm{span}\, B' = V$ & B' is now lin indep. Hence B' is a basis. $\square$

Theorem: Every fin dim vector space has a basis.

Proof: By def, every fin dim vector space has a spanning set. A basis is now obtained by reducing it as in the previous proposition. $\square$

We can also construct a basis by extending a lin indep list:

Prop: Every lin indep set of vectors in a fin. dim. vector space can be extended to a basis

Proof: If $u_1, \dots, u_m$ is linearly indep & $V = \mathrm{span}(w_1, \dots, w_n)$ then $V = \mathrm{span}(u_1, \dots, u_m, w_1, \dots, w_n)$. The reduction in the above prop now produces a basis for V, and it will not remove any of the u_i's because of lin independence. $\square$

Thm: If V is fin dim and $U \subset V$ is a subspace, then there exists a subspace $W \subset V$ such that $V = U \oplus W$.

Proof: V fin dim $\Rightarrow$ U fin dim.

So we pick a basis $u_1, \dots, u_m$ of U. Extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, w_1, \dots, w_n$ for V.

By construction $w_1, \dots, w_n$ is lin indep (else it wouldn't be part of a basis for V).

Define $W = \operatorname{span}(w_1, \dots, w_n)$. We now need to show

$V = U + W$ and $U \cap W = \{0\}$.

Any $v \in V$ can be written as

$$v = \underbrace{a_1 u_1 + \cdots + a_m u_m}_{\in U} + \underbrace{b_1 w_1 + \cdots + b_n w_n}_{\in W}$$

So $v = u + w \in U + W$ and so $V = U + W$ since $U + W \subset V$ is already known to be a subspace.

Next assume $\exists v \in U \cap W$ then write

$$v = a_1 u_1 + \cdots + a_m u_m$$

$$v = b_1 w_1 + \cdots + b_n w_n$$

for some $a_i, b_j \in \mathbb{F}$ $\forall i, j$. Then

$$0 = v - v = \sum_{i=1}^{m} a_i u_i - \sum_{j=1}^{n} b_j w_j$$

so $a_i = 0$ $\forall i$ and $b_j = 0$ $\forall j$ by lin indep of $u_1, \ldots, u_m, w_1, \ldots, w_n$

$\Rightarrow v = \{0\}$ so $U \cap W = \{0\}$ $\square$

§ Dimension

Prop: Any two bases of a fin dim vector space have the same length.

Proof: Let B_1 and B_2 be two bases.

B_1 lin indep & B_2 spans

$\Rightarrow$ length $B_1 \leq$ length B_2

But B_2 lin indep & B_1 spans

$\Rightarrow$ length $B_2 \leq$ length B_1

$\Rightarrow$ length $B_1 =$ length B_2 $\square$

Def: The dimension of a fin dim vector space V is defined as

dim V = length of any basis for V.

Ex: (a) $(1,0,\ldots,0),\ldots,(0,\ldots,0,1)$ Standard basis of $\mathbb{F}^n$ has length n so $\dim \mathbb{F}^n = n$

(b) $1, z, \ldots, z^m$ standard basis of $\mathcal{P}_m(\mathbb{F})$ has length $m+1$ so $\dim \mathcal{P}_m(\mathbb{F}) = m+1$.

Prop: If $U \subset V$ is a subspace of a fin dim vector space V, then $\dim U \leq \dim V$.

Proof: A previous result guarantees U is fin dim. Pick bases $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ of U and V.

$v_1, \ldots, v_n$ spans V & $u_1, \ldots, u_m$ is lin independent in V

$\Rightarrow \dim U = m \leq n = \dim V$ $\square$

Prop: V fin dim vector space.

Every lin indep list in V of length $\dim V$ is a basis.

<u>Proof</u>: A lin indep list can be extended to a basis. But if the list is already of length $\dim V$, no vectors are needed to be added & the lin indep list was already a basis to begin with. $\square$

<u>Prop</u>: If V is fin dim and $U \subset V$ such that $\dim U = \dim V$, then $U = V$.

<u>Proof</u>. If $u_1, \dots, u_m$ is a basis for U, its lin indep in V. & since $m = \dim U = \dim V$, it has the correct length (in V), so it's a basis in V, and

$$V = \operatorname{span}(u_1, \dots, u_m) = U$$

$\square$

Similar to the above, we also have:

<u>Prop</u>: If V is fin dim, and $V = \text{span}(v_1, \ldots, v_m)$ where $m = \dim V$, then $v_1, \ldots, v_m$ is a basis for V.

<u>Theorem</u>. Let V_1, V_2 be subspaces of a fin dim vector space V. Then

$$\dim V_1 + V_2 = \dim V_1 + \dim V_2 - \dim V_1 \cap V_2$$

<u>Proof</u>: Let $v_1, \ldots, v_m$ be a basis for $V_1 \cap V_2$. This list is lin indep in both V_1 and V_2, so we can find extensions to bases

$$v_1, \ldots, v_m, u_1, \ldots, u_n \quad \text{of } V_1$$
$$v_1, \ldots, v_m, w_1, \ldots, w_\ell \quad \text{of } V_2$$

So $\dim V_1 \cap V_2 = m$
$\dim V_1 = m+n$
$\dim V_2 = m+\ell$

We will show $v_1, \ldots, v_m, u_1, \ldots, u_n, w_1, \ldots, w_\ell$ is a basis for $V_1 + V_2$ which would conclude the proof.

It is obvious that

$$V_1+V_2 = \text{span}(v_1,\dots,v_m,u_1,\dots,u_n,w_1,\dots,w_\ell)$$

& so it suffices to show it's lin indep. Assume

$$\sum_{p=1}^{m} a_p v_p + \sum_{q=1}^{n} b_q u_q + \sum_{r=1}^{\ell} c_r w_r = 0. \quad (*)$$

Then

$$\sum_{r=1}^{\ell} c_r w_r = -\sum_{p=1}^{m} a_p v_p - \sum_{q=1}^{n} b_q u_q$$

By assumtion $\sum_{r=1}^{\ell} c_r w_r \in V_2$, but in this eqn RHS $\in V_1$, so

$$\sum_{r=1}^{\ell} c_r w_r \in V_1 \cap V_2 \Rightarrow$$

$$\sum_{r=1}^{\ell} c_r w_r = \sum_{s=1}^{m} d_s v_s \text{ for some } d_s \in \mathbb{F}$$

$$\Leftrightarrow \sum_{s=1}^{m} d_s v_s - \sum_{r=1}^{\ell} c_r w_r = 0$$

implies $d_s = 0\ \forall s$ and $c_r = 0\ \forall r$ because $v_1, \dots, v_m, w_1, \dots, w_\ell$ lin indep in V_2. Then (*) becomes

$$\sum_{p=1}^{m} a_p v_p + \sum_{q=1}^{n} b_q u_q = 0$$

Which implies $a_p = 0\ \forall p$ and $b_q = 0\ \forall q$ Since $v_1, \dots, v_m, u_1, \dots, u_n$ is lin indep in V_1. This shows $v_1, \dots, v_m, u_1, \dots, u_n, w_1, \dots, w_\ell$ is lin indep in $V_1 + V_2$ & hence a basis.

$\square$

So far we have studied individual vector spaces & their subspaces. We will now study maps between them.

§ Linear maps (3A)

As before $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and all vector spaces are defined over $\mathbb{F}$ unless stated otherwise.

Def: A *linear map* from V to W is a function $T : V \to W$ such that:

- (additivity) $T(u+v) = T(u) + T(v) \quad \forall u, v \in V$
- (homogeneity) $T(\lambda u) = \lambda T(u) \quad \forall \lambda \in \mathbb{F}, u \in V.$

We will denote the set of linear maps $V \to W$ by $\mathcal{L}(V, W)$. We also write $\mathcal{L}(V) := \mathcal{L}(V, V)$.

Ex: • The constant map

$$\begin{aligned} V &\longrightarrow W \\ v &\longmapsto 0 \end{aligned}$$ that sends every vector to 0

is called the zero map and is sometimes simply denoted by $0 \in \mathcal{L}(V,W)$.

• The linear map $\begin{aligned} V &\longrightarrow V \\ v &\longmapsto v \end{aligned}$ is called the identity map. It's usually denoted by $I \in \mathcal{L}(V)$.

• Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ by $Dp = p'$ for any $p \in \mathcal{P}(\mathbb{R})$. We know from calculus that $(p+q)' = p'+q'$ and $(\lambda p)' = \lambda p'$, $\lambda \in \mathbb{R}$, meaning that D is indeed linear.

• Similarly $Tp = \int_0^1 p(x)\,dx$ is a linear map $\mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}$.

• $T: \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$ defined by

$(Tp)(x) = x^2 p(x)$ is linear.

- Recall $\mathbb{F}^\infty = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{F} \ \forall i\}$.

  $$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

  is a linear map $\mathbb{F}^\infty \to \mathbb{F}^\infty$.

- $$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

  $$(x, y, z) \longmapsto (2x - y + 3z, 7x + 5y - 6z)$$

  is linear.

- $$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

  $(x, y) \mapsto x^2 + y^2$ is <u>not</u> linear

- Fix $q \in \mathcal{P}(\mathbb{R})$. Then

  $$\mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$$

  $$p \longmapsto p \circ q$$

  is linear.

<u>Prop</u> (linear map lemma)

Suppose $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are bases for V and W, respectively. Then there exists a unique linear map $T: V \to W$ such that $Tv_k = w_k \ \forall k$

Proof: Existence: For any $v \in V$ write $v = \sum_{i=1}^{n} c_i v_i$ & define

$$Tv = \sum_{i=1}^{n} c_i w_i .$$

We then have $Tv_k = w_k$ by def for any $k \in \{1, \dots, n\}$. To show T is linear, pick $u \in V$ & write $u = \sum_{i=1}^{n} b_i v_i$. Thus

$$u + v = \sum_{i=1}^{n} b_i v_i + \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} (b_i + c_i) v_i$$

so

$$T(u+v) = \sum_{i=1}^{n} (b_i + c_i) w_i = \sum_{i=1}^{n} b_i w_i + \sum_{i=1}^{n} c_i w_i$$

$$= Tu + Tv.$$

If $\lambda \in \mathbb{F}$ then

$$\lambda v = \lambda \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} (\lambda c_i) v_i , \text{ so}$$

$$T(\lambda v) = \sum_{i=1}^{n} (\lambda c_i) w_i = \lambda \sum_{i=1}^{n} c_i w_i = \lambda T v.$$

<u>Uniqueness</u>: If $S \in \mathcal{L}(V,W)$ is any other linear map w/ $S(v_k) = w_k \ \forall k$ we have

$$Tv = \sum_{i=1}^{n} c_i w_i = \sum_{i=1}^{n} c_i S(v_i)$$
$$= \sum_{i=1}^{n} S(c_i v_i) = S\left(\sum_{i=1}^{n} c_i v_i\right) = Sv$$

for any $v \in V$, so $T = S$. $\square$

It turns out that the set of linear maps $\mathcal{L}(V,W)$ is itself a vector space.

<u>Def</u>: Suppose $S, T \in \mathcal{L}(V,W)$. Then $S+T \in \mathcal{L}(V,W)$ and $\lambda T \in \mathcal{L}(U,W)$ for $\lambda \in \mathbb{F}$ are defined by

$$(S+T)(v) = S(v) + T(v)$$
$$(\lambda T)(v) = \lambda T(v). \qquad \forall v \in V.$$

Prop: $\mathcal{L}(V,W)$ is a vector space with the addition & scalar mult defined above.

Def: If $T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$ we may define their product $ST \in \mathcal{L}(U,W)$ to be the linear map

$$(ST)(u) = S(Tu).$$

Remark: ST is just usual composition of linear maps.

Prop. The product of linear maps satisfy the following properties:

- (associativity) $(T_1 T_2)T_3 = T_1(T_2 T_3)$ for any linear maps $V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \xrightarrow{T_3} V_4$.
- (identity) $TI_V = I_W T = T$ for any $T \in \mathcal{L}(V,W)$, where $I_V \in \mathcal{L}(V)$, $I_W \in \mathcal{L}(W)$ are the identity maps on V and W,

respectively.

- (distributive properties)

$$(S_1+S_2)T = S_1T + S_2T$$

$$S(T_1+T_2) = ST_1 + ST_2$$

$T, T_1, T_2 \in \mathcal{L}(U,V)$, $S, S_1, S_2 \in \mathcal{L}(V,W)$.

<u>Ex</u> The product of linear maps is not commutative. Recall $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$, $Dp = Dp'$ and $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ $(Tp)(x) = x^2 p(x)$ from before. Then

$$((DT)p)(x) = D(x^2 p(x)) = x^2 p'(x) + 2x p(x)$$

$$(TDp)(x) = (Tp')(x) = x^2 p'(x).$$

<u>Ex</u>: $T, S \in \mathcal{L}(\mathbb{F}^2)$ defined by

$$T(x,y) = (y,x), \quad S(x,y) = (x+y, y)$$

then

$(TS)(x,y) = T(x+y,y) = (y, x+y)$

$(ST)(x,y) = S(y,x) = (y+x, x)$

<u>Prop</u>: If $T \in \mathcal{L}(V,W)$ then $T(0)=0$.

<u>Proof</u>: $T(0) = T(0+0) = T(0)+T(0)$

Add $-T(0)$ to both sides:

$$0 = T(0). \qquad \square$$

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§ Null space & range (3B)

<u>Def</u>: Let $T \in \mathcal{L}(V,W)$. The <u>null space</u>, denoted by null T, is the subset of V

$$\text{null } T = \{ v \in V \mid Tv = 0 \}.$$

<u>Rem.</u>: The more common term of the null space is "kernel", and the notation $\ker T$.

Ex: • Let $0 \in \mathcal{L}(V,W)$ be the zero map. null $0 = V$ since everything maps to 0.

• Let $\varphi \in (\mathbb{R}^3, \mathbb{R})$ be def by $\varphi(x,y,z) = x+2y+3z$.

null $\varphi = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y+3z = 0\}$

this is a plane in $\mathbb{R}^3$

• Consider $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$. Then

null $D = \{p \in \mathcal{P}(\mathbb{R}) \mid p' = 0\}$

$= \{$constant polynomials$\} = \mathcal{P}_0(\mathbb{R})$.

• $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$, $(Tp)(x) = x^2 p(x)$,

null $T = \{p \in \mathcal{P}(\mathbb{R}) \mid x^2 p(x) = 0 \ \forall x \in \mathbb{R}\}$.

$= \{0\}$

• $T \in \mathcal{L}(\mathbb{F}^\infty)$, $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$

null $T = \{(x_1, x_2, \dots) \in \mathbb{F}^\infty \mid (x_2, x_3, \dots) = 0\}$

$= \{(x_1, 0, 0, \dots) \mid x_1 \in \mathbb{F}\}$.