MATH 435 SUPPLEMENT: PARAMETERS

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Definition 1. Fix an interval $(\alpha, \beta) \subset \mathbb{R}$. A parameter u on (α, β) is by definition a smooth bijection with smooth inverse to some other interval $(\tilde{\alpha}, \tilde{\beta})$; i.e., $u\colon (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is a smooth bijection, and its inverse $u^{-1}\colon (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ is also smooth.

In class, we proved:

Proposition 2. A smooth function u on (α, β) is a parameter if and only if its derivative $\dot{u}$ is non-vanishing, i.e., $\dot{u}(t) \neq 0$ for all $t \in (\alpha, \beta)$.

More precisely:

(i) If $u\colon (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is a parameter, then $\dot{u}$ is non-vanishing.
(ii) If $u\colon (\alpha, \beta) \to \mathbb{R}$ is a smooth function with $\dot{u}$ non-vanishing, then the image of u is an open interval $(\tilde{\alpha}, \tilde{\beta})$ and $u\colon (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is a parameter.

In terms of the notion of parameter, the notion of "reparametrization" (Definition 1.3.1 in the book) can now be stated as: $\tilde{\gamma}$ is a reparametrization of γ if $\gamma = \tilde{\gamma} \circ u$ for some parameter u. Let us give this a name:

Definition 3. Given a curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ and a parameter $u\colon (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$, we call the function $\tilde{\gamma} = \gamma \circ u^{-1}\colon (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ the reparametrization of γ with parameter u.

The main point of introducing the notion of parameter is the following definition:

Definition 4. Given a parameter u on (α, β) and a curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ (or a function $(\alpha, \beta) \to \mathbb{R}$, which is just the special case $n = 1$), we define the derivative of γ with respect to u to be the function $\frac{\mathrm{d}\gamma}{\mathrm{d}u}\colon (\alpha, \beta) \to \mathbb{R}^n$ defined by

$$\frac{\mathrm{d}\gamma}{\mathrm{d}u}(t) = \frac{\dot{\gamma}(t)}{\dot{u}(t)}. \tag{1}$$

We then define $\frac{\mathrm{d}^2\gamma}{\mathrm{d}u^2} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}u}\right)$, and so on.

This definition is motivated by the idea that the derivative with respect to u should satisfy the chain rule:

$$\dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}\gamma}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}\gamma}{\mathrm{d}u} \cdot \dot{u},$$

and rearranging gives (1).

Remark 5. In fact, it follows from the chain rule that if $\tilde{\gamma}$ is the reparametrization with parameter u, then $\frac{\mathrm{d}\gamma}{\mathrm{d}u}(t) = \dot{\tilde{\gamma}}(u(t))$. The point of the definition, then, is that it allows us to avoid dealing with the reparametrization $\tilde{\gamma}$.

(By induction, one can also show that $\frac{\mathrm{d}^k\gamma}{\mathrm{d}u^k} = \tilde{\gamma}^{(k)} \circ u$ for any k.)

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Example 6. The most important example of a parameter is the arc-length: it is easy to prove (and is proven in Proposition 1.3.6 in the book) that for any *regular* curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$, the arc-length s starting at any $t_0 \in (\alpha, \beta)$ is a parameter.

This parameter has the special property that $\|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\| = 1$, i.e., $\|\frac{\mathrm{d}\gamma}{\mathrm{d}s}(t)\| = 1$ for every $t \in (\alpha, \beta)$. Indeed, we showed in class that $\dot{s}(t) = \|\dot{\gamma}(t)\|$, and hence by definition

$$\left\|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right\| = \left\|\frac{\dot{\gamma}}{\dot{s}}\right\| = \left\|\frac{\dot{\gamma}}{\|\dot{\gamma}\|}\right\| = 1.$$

Let us give a name to such parameters.

Definition 7. Given a curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$, a parameter u on (α, β) is called a unit-speed parameter for γ if $\|\frac{\mathrm{d}\gamma}{\mathrm{d}u}\| = 1$.

It follows immediately from Remark 5 that u is a unit-speed parameter for γ if and only if the reparametrization $\tilde{\gamma}$ with parameter u is a unit-speed curve.

Corollary 1.3.7 in the book shows that there are not many unit-speed parameters:

Proposition 8. If s is the arc-length of the curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ starting at some $t_0 \in (\alpha, \beta)$, then for any unit-speed parameter u for γ, we have

$$u = \pm s + c$$

for some constant $c \in \mathbb{R}$.