MATH 435 SUPPLEMENT: SOME TOPOLOGY

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Here are some useful facts about open subsets of $\mathbb{R}^n$:

Proposition 1.

(i) If $U, V \subset \mathbb{R}^n$ are open sets, then $U \cap V$ is open.

(ii) If $\{U_i\}_{i \in I}$ is an arbitrary collection of open subsets $U_i \subset \mathbb{R}^n$ (here, I is an arbitrary indexing set), then the union

$$\bigcup_{i \in I} U_i := \{x \in \mathbb{R}^n \mid x \in U_i \text{ for some } i \in I\}$$

is also open.

(iii) The empty set $\emptyset$ and $\mathbb{R}^n$ itself are both open.

The proof is fairly straightforward, using the definition open set.

Definition 2. Any collection of subsets of a set X satisfying conditions (i)-(iii) of the above proposition is called a topology on X.

Thus, we can say that the open subsets of $\mathbb{R}^n$ form a topology on $\mathbb{R}^n$.

Now let $X \subset \mathbb{R}^n$ be any subset.

Definition 3. A set $V \subset X$ is open in X if it is of the form $X \cap W$ for some open set $W \subset \mathbb{R}^n$.

Proposition 4. The open subsets of X form a topology on X.

Definition 5. Give a point $\mathbf{x} \in X$ and a real number $r > 0$, we define the ball of radius r in X centered at $\mathbf{x}$, denoted $\mathrm{D}_r^X(\mathbf{x})$ or $\mathrm{B}_r^X(\mathbf{x})$ to be the set

$$\mathrm{D}_r^X(\mathbf{x}) = \{\mathbf{y} \in X \mid \|\mathbf{x} - \mathbf{y}\| < r\}.$$

Equivalently, it is the intersection $\mathrm{D}_r^X(\mathbf{x}) = \mathrm{D}_r(\mathbf{x}) \cap X$.

Proposition 6. A subset $V \subset X$ is open if and only if for each $\mathbf{x} \in V$, there is some $r > 0$ such that $\mathrm{D}_r^X(\mathbf{x}) \subset V$.

Proof. The ($\Rightarrow$) is a homework problem.

For the other direction, suppose that for each $\mathbf{x} \in V$, we have some $r_{\mathbf{x}} > 0$ such that $\mathrm{D}_{r_{\mathbf{x}}}(\mathbf{x}) \cap X = \mathrm{D}_{r_{\mathbf{x}}}^X(\mathbf{x}) \subset V$. We need to find an open set $W \subset \mathbb{R}^n$ such that $V = X \cap W$.

We define W to be the union

$$W = \bigcup_{\mathbf{x} \in V} \mathrm{D}_{r_{\mathbf{x}}}(\mathbf{x}),$$

which is open, being a union of open sets. It remains to verify that $V = W \cap X$.

We have $V \cap X$ by definition, and also $V \cap W$, since for each $\mathbf{x}$, we have $\mathbf{x} \in \mathrm{D}_{r_{\mathbf{x}}}(\mathbf{x}) \subset W$. Hence $V \subset W \cap X$.

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On the other hand, we have

$$W \cap X = \left(\bigcup_{\mathbf{x} \in V} \mathrm{D}_{r_{\mathbf{x}}}(\mathbf{x}) \right) \cap X = \bigcup_{\mathbf{x} \in V} \big(\mathrm{D}_{r_{\mathbf{x}}}(\mathbf{x}) \cap X \big) = \bigcup_{\mathbf{x} \in V} \mathrm{D}^X_{r_{\mathbf{x}}}(\mathbf{x}) \subset V,$$

where in the second equality, we use a general property of unions and intersections.

Hence $V = W \cap X$, as desired. □

Proposition 7. If $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ and $f \colon X \to Y$ is continuous, then for any $V \subset Y$ which is open in Y, the *preimage of* V *under* f

$$f^{-1}(V) := \{\mathbf{x} \in X \mid f(\mathbf{x}) \in V\}$$

is open in X.

This is a homework problem.

In fact, the converse is true as well: if the preimage of each open set under f is open, then f is continuous.