Homework 2

Due: Wednesday, January 24

Math 435, Fall 2024

Problem 1. Spivak 3-1

Problems. **3-1.** Let $f\colon [0,1]\times[0,1]\to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Show that f is integrable and $\int_{[0,1]\times[0,1]} f = \frac{1}{2}$.

Problem 2. Spivak 3-2

3-2. Let $f\colon A \to \mathbf{R}$ be integrable and let $g = f$ except at finitely many points. Show that g is integrable and $\int_A f = \int_A g$.

Problem 3. Spivak 3-3, part (a)

3-3. Let $f,g\colon A \to \mathbf{R}$ be integrable.

(a) For any partition P of A and subrectangle S, show that

$$m_S(f) + m_S(g) \le m_S(f+g) \quad \text{and} \quad M_S(f+g) \le M_S(f) + M_S(g)$$

and therefore

$$L(f,P) + L(g,P) \le L(f+g,P) \quad \text{and} \quad U(f+g,P) \le U(f,P) + U(g,P).$$

Problem 4. Spivak 3-5

3-5. Let $f,g\colon A \to \mathbf{R}$ be integrable and suppose $f \le g$. Show that $\int_A f \le \int_A g$.

Problem 5. Spivak 3-6

3-6. If $f\colon A \to \mathbf{R}$ is integrable, show that $|f|$ is integrable and $\left|\int_A f\right| \le \int_A |f|$.

Hint: One approach is to first show that if f is integrable, then so is $\max(0,f)$, and $\int_A f \le \int_A \max(0,f)$; then use that $|f| = |-f| = \max(0,f) + \max(0,-f)$ and apply the results of the previous problems.