Math 435: Lecture 42

April 26, 2024

Reference: Tapp, pp. Chapters 4 and 5

Topics:

- Examples of Gauss-Bonnet
  - For the unit sphere S^2, we have constant curvature $K = 1$, and area 4π, hence we find that $\chi(\mathrm{S}^2) = 2$.
  - We can also compute it directly: for example, if we triangulate S^2 as a tetrahedron, we obtain $\chi = 4 - 6 + 4 = 2$, or if as an octahedron, we obtain $\chi = 8 - 12 + 6 = 2$.
  - A useful fact is that you can actually compute the Euler characteristic using *any* partition into polygons, not just triangles. The reason is that you can always subdivide a polygon into triangles, and it won't change the Euler characteristic, since each time we connect two corners, we are adding both a face and an edge.
  - Hence, we can also compute $\chi(\mathrm{S}^2)$ with a cube $6 - 12 + 8 = 2$.
  - Next, for the *torus*, it is harder to compute the total Gaussian curvature directly. But we can triangulate (or "polygonate") it using a single face, two edges, and one vertex, hence $\xi = 1 - 2 + 1 = 0$. Thus the *total Gaussian curvature of a torus is 0.*
  - There is a second way to see this, which is by attaching a *handle* to the sphere.
  - That is, we consider the cylinder $\{(x, y, z) \mid x^2 + y^2 = 1 \text{ and } z \in [0, 1]\}$. It has a polygonation with 1 face, 3 edges, and 2 vertices (hence $\chi = 0$).
  - We can now start with a polygonated surface, remove two faces, and glue on a cylinder along the two resulting boundaries; altogether, we will have removed two faces, and added one face and one edge, hence decreased χ by 2.
  - If we add a handle to S^2, we obtain a torus, so we see again that χ of the torus is 2.
  - In general, the *genus* $g(S)$ of a compact surface $S \subset \mathbb{R}^3$ is its number of handles (equivalently, the number of "cuts" that need to be made in order to turn it into a sphere); hence a surface has genus 0, a torus has genus 1, and if we attach another handle, we get a surface of genus 2 (a kind of "double torus").
  - Since $\chi(\mathrm{S}^2) = 2$, we see in general that the Euler characteristic of a genus g surface is
$$\chi = 2 - 2g.$$
  - Note that besides the two special cases $g = 0$ (the sphere) and $g = 1$ (the sphere) the *total Gaussian curvature is always negative.*
- Gauss-Bonnet with boundary
  - The proof of Gauss-Bonnet involves introducing two generalizations of it, which are interesting in their own right.
  - The first concerns surfaces with boundary, and says that if S is a surface with boundary, then
$$\int_S K + \int_{\partial S} \kappa_{\mathrm{g}} = 2\pi\chi(S). \tag{4}$$
(Note that κ_{g} in principle depends on an orientation of S, but switching the orientation changes both the direction of the induced orientation on the boundary curve, and the sign of κ_{g}, so these will cancel out.)
  - Note that if $\partial S = \emptyset$, this recovers the old formula.

  - In particular, this shows that the Euler characteristic of a surface *with boundary* is also independent of the triangulation (which we didn't know yet).
  - The second variant is about a *polygonal region* R on a surface S – i.e., it is the case where we have not only boundaries, but corners. In this case, the formula is:

$$\int_R K + \int_{\partial R} \kappa_{\mathrm{g}} + \sum_i \alpha_i = 2\pi \tag{5}$$

    where the sum $\sum_i \alpha_i$ is over the external angles at the corners of the polygon. Note that we might also write the right side as $2\pi\chi(R)$ since the Euler characteristic of a disk is $1 - 1 + 1 = 1$.
  - Compare this with the Umlaufsatz: $\int_a^b \kappa_{\mathrm{s}}(t) + \sum_{i=1}^n \alpha_i = 2\pi$.
  - (As stated, this version isn't quite a *generalization* of (4), since a whole surface will not in general be a polygonal region, but one can formulate a more general version of the theorem which really is a generalization of (4), see Tapp Theorem 6.8.)
- Geodesic polygons
  - Though the polygonal version (5) of Gauss-Bonnet will mainly be a tool for us to prove the version (4) for a whole surface, it is actually very interesting in its own right, especially in the case of *geodesic polygons*, i.e., polygons whose edges are geodesics, so that the term κ_{g} vanishes.
  - In the plane, where $K = 0$, it then just reduces to the formula $\sum_i \alpha_i = 2\pi$, or the equivalent version with internal angles $\sum \beta_i = (n-2)\pi$.
  - Another interesting example to consider is that of *spherical geometry*, i.e., the study of figures formed out of great circles on the sphere.
  - Here, we see, for example, that the usual angle sum of π for a triangle is increased (the triangles are "fatter"), and moreover the defect $\int_R K$ is proportional to the *area* of the triangle.
    (In particular, *very small* triangles have angle sum nearly π, as on a small scale, any surface is "nearly flat".)
  - There is a third famous case – called *hyperbolic geometry* – which is that of a surface with constant negative rather than positive Gaussian curvature, $K = -1$.
  - It is difficult to come across such surfaces in $\mathbb{R}^3$, but there is a famous "abstract" surface with this property called the *hyperbolic plane*, which we do not have time to explain now.
  - In any case, here, the triangles are instead *thinner*, and the defect again is proportional to the area of the triangle.
  - The dutch artist M.C. Escher made some famous and beautiful artistic renditions of the hyperbolic plane, which you should look at.
- Sketch of the proof
  - Proving the Gauss-Bonnet theorem has two steps: first prove (5), and then use that to prove (4).
  - The proof of (5) is the more difficult but (somewhat) less interesting part. There are two basic ideas involved: (i) use Stokes' theorem to relate the integral $\int_R K$ over the interior to the integral $\int \kappa_{\mathrm{g}}$ over the boundary, and (ii) use a variant of the argument from the Umlaufsatz (including the part about smoothing the corners) to relate the integral $\int \kappa_{\mathrm{g}}$ to a "total change of angle", which again comes out to be 2π.

- The fun part of the proof is deducing the Gauss-Bonnet theorem (4) for a whole surface, assuming the version (5) about polygonal regions.
- Choose a finite triangulation $T_1, \dots T_F$ of the given surface(-with-boundary) S.
- By assumption the equation (5) holds with $R = T_i$ for each i.
- Let us sum both sides of this equation over all i. On the right-hand side, we just get $2\pi F$.
- The first term on the left-hand side simply adds up to $\int_S K$.
- The sum of the terms $\int_{\partial T_i} \kappa_{\mathrm{g}}$ give the sum over all the edges e of all the triangles of $\int_e \kappa_{\mathrm{g}}$. However, each *interior* edge (i.e., edge which is not on a boundary) is counted twice with opposite signs (because it has the opposite induced orientation from the two adjacent triangles), hence these cancel out. We are thus left with $\int_{\partial S} \kappa_{\mathrm{g}}$.
- Finally, the terms $\sum_i \alpha_i$ add up to the sum of all the external angles of all the vertices of all the triangles.
- Recall that the external angle α_i is $\pi - \beta_i$, where β_i is the internal angle.
- Hence, we can write this as $A\pi - \sum \beta_i$, where A is the total number of angles, and the second term is the sum of all the *internal* angles (at all vertices).
- Now the sum of all the internal angles at an *interior* vertex is 2π, whereas at an *exterior* vertex (i.e., one lying on ∂S), it is just π. Hence $\sum \beta_i = 2\pi V_{\mathrm{int}} + \pi V_{\mathrm{ext}}$.
- On the other hand, the number angles at a given interior vertex v is simply the number of edges E_v emanating from V, whereas at an exterior vertex it is $E_v - 1$. Hence, since each edge occurs at two vertices, summing them all up gives $A = 2E - V_{\mathrm{ext}}$.
- Hence, we have $A\pi - \sum \beta_i = (2E - V_{\mathrm{ext}})\pi - (2\pi V_{\mathrm{int}} + V_{\mathrm{ext}}) = 2\pi(E - V)$.
- Adding everything up, we thus have
$$\int_S K + \int_{\partial S} \kappa_{\mathrm{g}} + 2\pi(E - V) = 2\pi F,$$
which is the Gauss-Bonnet formula.