Homework 1

Due: Wednesday, January 17

Math 435, Fall 2024

Problem 1. Spivak 1-10

1-10.* If $T: \mathbf{R}^m \to \mathbf{R}^n$ is a linear transformation, show that there is a number M such that $|T(h)| \leq M|h|$ for $h \in \mathbf{R}^m$. *Hint:* Estimate $|T(h)|$ in terms of $|h|$ and the entries in the matrix of T.

Hint: Write each component of $T(h)$ as an inner product $\langle a_i, h\rangle$ for an appropriate vector $a_i \in \mathbb{R}^m$ and use the Cauchy-Schwarz inequality.

Problem 2. Spivak 1-14

Problems. **1-14.*** Prove that the union of any (even infinite) number of open sets is open. Prove that the intersection of two (and hence of finitely many) open sets is open. Give a counterexample for infinitely many open sets.

Problem 3. Spivak 1-16

1-16. Find the interior, exterior, and boundary of the sets

$$\{x \in \mathbf{R}^n: |x| \leq 1\}$$
$$\{x \in \mathbf{R}^n: |x| = 1\}$$
$$\{x \in \mathbf{R}^n: \text{each } x^i \text{ is rational}\}.$$

Hint: In this and the following problem, you may take for granted the following fact: every open interval in $\mathbb{R}$ contains a rational number.

Problem 4. Spivak 1-19

1-19.* If A is a closed set that contains every rational number $r \in [0,1]$, show that $[0,1] \subset A$.

Problem 5. Spivak 1-24

1-24. Prove that $f: A \to \mathbf{R}^m$ is continuous at a if and only if each f^i is.

Problem 6. Spivak 1-29

1-29. If A is compact, prove that every continuous function $f: A \to \mathbf{R}$ takes on a maximum and a minimum value.

Hint: You may take for granted the following fact: every closed and bounded subset of $\mathbb{R}$ has a maximal and minimal element.

Problem 7. Spivak 2-1

Problems. **2-1.*** Prove that if $f: \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^n$, then it is continuous at a. *Hint:* Use Problem 1-10.

Problem 8. Spivak 2-29

2-29. Let $f: \mathbf{R}^n \to \mathbf{R}$. For $x \in \mathbf{R}^n$, the limit

$$\lim_{t \to 0} \frac{f(a + tx) - f(a)}{t},$$

if it exists, is denoted $D_x f(a)$, and called the **directional derivative** of f at a, in the direction x.

(a) Show that $D_{e_i} f(a) = D_i f(a)$.

(b) Show that $D_{tx} f(a) = t D_x f(a)$.

(c) If f is differentiable at a, show that $D_x f(a) = Df(a)(x)$ and therefore $D_{x+y} f(a) = D_x f(a) + D_y f(a)$.