Homework 5

Due: Wednesday, February 14

Math 435, Fall 2024

Problem 1. Let V and W be vector spaces.

(a) Prove that the operation $\otimes\colon \mathcal{T}^k \times \mathcal{T}^l \to \mathcal{T}^{k+l}$ is bilinear, i.e., given k-tensors $S, S' \in \mathcal{T}^k$, l-tensors $T, T' \in \mathcal{T}^l$, and $a \in \mathbb{R}$, we have:

$$(S+S')\otimes T = S\otimes T + S'\otimes T \qquad S\otimes(T+T') = S\otimes T + S\otimes T' \qquad (aS)\otimes T = a(S\otimes T) = S\otimes(aT)$$

(b) Prove that $\otimes$ is associative, i.e., for any k-tensor S, l-tensors T, and m-tensor U, we have $(S\otimes T)\otimes U = S\otimes(T\otimes U)$

(c) Prove that for any linear map $f\colon V\to W$ and tensors S and T on W, we have $f^*(S\otimes T) = f^*S\otimes f^*T$.

Problem 2. Let V and W be vector spaces.

(a) Prove that $\wedge\colon \Lambda^k(V)\times\Lambda^l(V)\to\Lambda^{k+l}(V)$ is bilinear

(b) Prove that for $\wedge\colon \Lambda^k(V)$ and $\Lambda^l(V)$, we have $\omega\wedge\eta = (-1)^{kl}\eta\wedge\omega$.
Hint: Show that there is a permutation $\tau\in \mathrm{S}_{k+l}$ such that for any $v_1,\ldots,v_{k+l}\in V$, we have

$$\omega(v_1,\ldots,v_k)\cdot\eta(v_{l+1},\ldots,v_{l+k}) = \eta(v_{\tau(1)},\ldots,v_{\tau(l)})\cdot\omega(v_{\tau(l+1)},\ldots,v_{\tau(l+k)}),$$

and hence, given any permutation $\sigma\in\mathrm{S}_{k+l}$, we have

$$\begin{aligned}\omega(v_{\sigma(1)},\ldots,v_{\sigma(k)})\cdot\eta(v_{\sigma(l+1)},\ldots,v_{\sigma(l+k)}) &= \eta(v_{\tau(\sigma(1))},\ldots,v_{(\tau(\sigma(l))})\cdot\omega(v_{(\tau(\sigma(l+1))},\ldots,v_{(\tau(\sigma(l+k))}),\\ &= \eta(v_{(\tau\circ\sigma)(1))},\ldots,v_{(\tau\circ\sigma)(l))})\cdot\omega(v_{(\tau\circ\sigma)(l+1)},\ldots,v_{(\tau\circ\sigma)(l+k)}),\end{aligned}$$

where $\tau\circ\sigma$ is the composite permutation. What is $\operatorname{sgn}\tau$?
Then recall that composing with τ (i.e., the operation $\sigma\mapsto\tau\circ\sigma$) is a bijection (with inverse $\sigma\mapsto\tau^{-1}\circ\sigma$), hence $\sum_{\sigma\in\mathrm{S}_{k+l}} a_\sigma = \sum_{\sigma\in\mathrm{S}_{k+l}} a_{\tau\circ\sigma}$ for any collection $\{a_\sigma\}_{\sigma\in\mathrm{S}_{k+l}}$.

(c) Prove that for any linear map $f\colon V\to W$ and alternating tensors ω and η on W, we have $f^*(\omega\wedge\eta) = f^*(\omega)\wedge f^*(\eta)$.

Problem 3. Spivak 4-1

Problems. **4-1.*** Let $e_1,\ldots,e_n$ be the usual basis of $\mathbf{R}^n$ and let $\varphi_1,\ldots,\varphi_n$ be the dual basis.

(a) Show that $\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}\,(e_{i_1},\ldots,e_{i_k}) = 1$. What would the right side be if the factor $(k+l)!/k!l!$ did not appear in the definition of $\wedge$?

(b) Show that $\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}(v_1,\ldots,v_k)$ is the determinant of the $k\times k$ minor of $\begin{pmatrix} v_1\\ \cdot\\ \cdot\\ \cdot\\ v_k\end{pmatrix}$ obtained by selecting columns $i_1,\ldots,i_k$.

Problem 4. Show that for any vector space V and any $\varphi_1,\ldots,\varphi_k\in V^*$ and any $v_1,\ldots,v_k\in V$, we have $\varphi_1\wedge\cdots\wedge\varphi_k(v_1,\ldots,v_k) = \det\left([\varphi_i(v_j)]_{i,j=1}^k\right)$.