Homework 11
Due: Wednesday, April 10
Math 435, Fall 2024

Problem 1. Tapp 1.23

EXERCISE 1.23. Let $\boldsymbol{\gamma}: I \to \mathbb{R}^3$ be a regular space curve, and let $P \subset \mathbb{R}^3$ be a plane that does not intersect the image of $\boldsymbol{\gamma}$. If $\boldsymbol{\gamma}$ comes closest to P at time t_0, prove that $\boldsymbol{\gamma}'(t_0)$ is parallel to P.

Recall that the distance from a point $\mathbf{p}$ to a plane P is by definition the distance from $\mathbf{p}$ to the closest point to $\mathbf{p}$ on P. If $\vec{n}$ is a normal vector to P and $\mathbf{q} \in P$ is any point on P, then the distance is $|\text{Proj}_{\vec{n}}(\mathbf{p} - \mathbf{q})|$.
If $P = \{(x, y, z) \mid ax + by + cz = d\}$, then (a, b, c) is a normal vector, and if $P = \{\mathbf{q} + s\vec{u} + t\vec{v} \mid s, t \in \mathbb{R}\}$, then $\vec{u} \times \vec{v}$ is a normal vector.

Problem 2. Tapp 1.29

EXERCISE 1.29. Consider the following pair of plane curves:

$$\boldsymbol{\gamma}(s) = (\cos s, \sin s), \quad s \in (-\pi, \pi),$$
$$\tilde{\boldsymbol{\gamma}}(t) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}\right), \quad t \in \mathbb{R}.$$

Verify that $\tilde{\boldsymbol{\gamma}}$ is a reparametrization of $\boldsymbol{\gamma}$. *HINT:* $t = \tan(s/2)$.

Problem 3. Tapp 1.36

EXERCISE 1.36. *Prove or disprove:* For a regular parametrized curve $\boldsymbol{\gamma}$ in $\mathbb{R}^n$, the measurement $f(t) = \frac{|\boldsymbol{\gamma}''(t)|}{|\boldsymbol{\gamma}'(t)|^2}$ is independent of parametrization.

Problem 4. Show that the curvature of a regular curve is constantly zero if and only if its trace lies on a line.

Problem 5. Tapp 1.39

EXERCISE 1.39. For constants $a, b, c > 0$, consider the "generalized helix" defined as $\boldsymbol{\gamma}(t) = (a \cos t, b \sin t, ct)$, $t \in \mathbb{R}$. Where is the curvature maximal and minimal?

Problem 6. Tapp 1.43

EXERCISE 1.43. Let $\boldsymbol{\gamma}: I \to \mathbb{R}^n$ be a regular curve. Assume that the function $t \mapsto |\boldsymbol{\gamma}(t)|$ has a local maximum value of r occurring at time t_0. Prove that

$$\kappa(t_0) \geq \frac{1}{r}.$$

Is there any *upper* bound for $\kappa(t_0)$ under these conditions?