Homework 12
Due: Wednesday, April 17
Math 435, Fall 2024

Problem 1. Tapp 1.47

> EXERCISE 1.47. Prove that the trace of every regular plane curve with constant nonzero signed curvature must equal a circle or a segment of a circle.

Problem 2. Find a plane curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ whose angle function is $\theta(t) = 2t$.

Problem 3. Tapp 1.56 **in the case where m and n have opposite parities** (i.e., one is even, an one is odd).

> EXERCISE 1.56. Let m, n be positive integers. Find the rotation index of the **Lissajous curve** $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ (see Fig. 1.26) defined as
>
> $$\boldsymbol{\gamma}(t) = (\cos(mt), \sin(nt)).$$

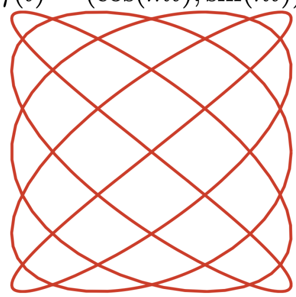

FIGURE 1.26. The Lissajous curve $\boldsymbol{\gamma}(t) = (\cos(5t), \sin(4t))$, $t \in [0, 2\pi]$

Again, you only need to consider the case where one of m and n is even and one is odd.
Hint: What happens to the γ when you reflect it across the x or y axis? What happens to its signed curvature?

Problem 4. Tapp 1.62

> EXERCISE 1.62. Calculate the curvature function and torsion function for the curve $\boldsymbol{\gamma}(t) = \left(t, t^2, t^3\right)$, $t \in \mathbb{R}$.

Hint: First prove the formula $\tau = \frac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{|\gamma' \times \gamma''|^2}$ from Tapp Exercise 1.65.

Problem 5. Compute the torsion function of the helix $\gamma(t) = (\cos t, \sin t, t)$, $t \in \mathbb{R}$.

Problem 6. Tapp 1.68

> EXERCISE 1.68. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a space curve, and let $t_0 \in I$ with $\kappa(t_0) \neq 0$. Let P denote the osculating plane at t_0 (translated to $\boldsymbol{\gamma}(t_0)$). For $t \in I$ near t_0, let $\boldsymbol{\beta}(t)$ denote the point of P closest to $\boldsymbol{\gamma}(t)$. Prove that $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ have the same curvature at time t_0.

Hint: Find an explicit formula for $\beta(t)$; note that $\vec{b}(t_0)$ is a normal vector to P.

Problem 7. Tapp 1.69

> EXERCISE 1.69. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a regular space curve (possibly with points where $\kappa = 0$ and hence where τ is undefined). *Prove or disprove*:
> (1) If the trace of $\boldsymbol{\gamma}$ lies in a plane, then τ equals zero everywhere it is defined.
> (2) If τ equals zero everywhere it is defined, then the trace of $\boldsymbol{\gamma}$ lies in a plane.