Homework 13
Due: Wednesday, April 24
Math 435, Fall 2024

Problem 1. Tapp 3.81

> EXERCISE 3.81. Let $f : S_1 \to S_2$ be a diffeomorphism between regular surfaces. Prove that f is an isometry if and only if for every regular curve $\gamma : [a, b] \to S_1$, the length of γ equals the length of $f \circ \gamma$.

Problem 2. Tapp 3.101

> EXERCISE 3.101. Prove that the following are equivalent for a diffeomorphism $f : S \to \tilde{S}$ between regular surfaces:
>
> (1) f is an isometry.
> (2) For every surface patch $\sigma : U \subset \mathbb{R}^2 \to V \subset S$, the first fundamental form of σ equals the first fundamental form of $f \circ \sigma$.
> (3) Every $p \in S$ is covered by a surface patch σ such that the first fundamental form of σ equals the first fundamental form of $f \circ \sigma$.

In (2) and (3), what is meant is that the functions E, F, G are the same for σ and for $f \circ \sigma$.

Problem 3. Let $\gamma = (\gamma_1, \gamma_2)\colon [a, b] \to \mathbb{R}^2$ be a simple closed curve and let $S \subset \mathbb{R}^3$ be the surface

$$S = \{(\gamma_1(u), \gamma_2(u), v) \mid u \in [a, b],\ v \in \mathbb{R}\}.$$

Prove that S is isometric to the standard cylinder $C = \{(x, y, z) \mid x^2 + y^2 = R^2\}$ of radius R for some R.

Problem 4. Tapp 3.103

> EXERCISE 3.103. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a helix of the form $\gamma(\theta) = (\cos\theta, \sin\theta, c\theta)$, where $c \neq 0$ is a constant, shown green in Fig. 3.39. For each value of θ, consider the infinite line (shown red) through $\gamma(\theta)$ that is parallel to the xy-plane and intersects the z-axis. The union of all these lines is called a **helicoid**, visualized as the surface swept out by the propeller of a rising helicopter (or lowering if $c < 0$). It can be covered by the single surface patch
>
> $$\sigma(\theta, t) = (t\cos\theta, t\sin\theta, c\theta), \quad t, \theta \in (-\infty, \infty).$$
>
> (1) Describe the first fundamental form in these coordinates.
> (2) What is the area of the portion of the helicoid corresponding to $0 < t < 1$ and $0 < \theta < 4\pi$?
> (3) At a point p of the helicoid, how does the angle that a unit normal vector at p makes with the z-axis depend on the distance of p to the z-axis?

For (3), use the unit normal with *positive* z-component, and in particular, answer: (i) does the angle increase or decrease as the distance from the z-axis grows, (ii) what is the limiting angle as the distance goes to 0 or ∞?

Problem 5. Let $\sigma\colon U \to V \subset S$ be a surface patch on a surface S with first fundamental form $E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2$. Prove that σ is angle-preserving (i.e., $\angle(\vec{u}, \vec{v}) = \angle(\mathrm{d}\sigma_{\mathbf{p}}(\vec{u}), \mathrm{d}\sigma_{\mathbf{p}}(\vec{v}))$ for all $\mathbf{p} \in U$ and $\vec{u}, \vec{v} \in \mathrm{T}_{\mathbf{p}}U = \mathbb{R}^2$) if and only if $E = G$ and $F = 0$.
Hint: You may want to first prove that if $T\colon X \to Y$ is a linear map between two-dimensional inner product spaces, then T is angle-preserving if and only if there is some constant C such that $\langle T\vec{u}, T\vec{v}\rangle = C\langle \vec{u}, \vec{v}\rangle$ for all $\vec{u}, \vec{v} \in X$. For the ($\Rightarrow$) direction, choose an orthonormal basis $\vec{b}_1, \vec{b}_2$ for X, and first prove that $|T\vec{b}_1| = |T\vec{b}_2|$ by showing that otherwise, $\angle(\vec{b}_1, \vec{b}_1 + \vec{b}_2) \neq \angle(T(\vec{b}_1), T(\vec{b}_1 + \vec{b}_2))$. Setting $c := |T\vec{b}_1| = |T\vec{b}_2|$, conclude from this that $|T\vec{v}| = c \cdot |\vec{v}|$ for all $\vec{v} \in X$, and thence that $\langle T\vec{u}, T\vec{v}\rangle = c^2\langle \vec{u}, \vec{v}\rangle$ for all $\vec{u}, \vec{v} \in X$.