Math 435: Lecture 15

February 12, 2024

Reference: Spivak, pp. 86-90

Topics:

- Orientations
  - By the theorem, any non-zero $\omega \in \Lambda^n(V)$ satisfies $\omega(v_1, \ldots, v_n)$ for any basis $v_1, \ldots, v_n$ (since by definition we must have $\omega(w_1, \ldots, w_n)$ for *some* vectors $w_1, \ldots, w_n$).
  - Thus, the set of bases of V is split into two subsets: those with $\omega(v_1, \ldots, v_n) > 0$ and those with $\omega(v_1, \ldots, v_n) < 0$.
  - Given two bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ with $w_i \sum_j a_{ij} v_j$, they will be in the same subset if and only if $\det(a_{ij}) > 0$.
  - This condition is independent of ω and always separates the bases of V into two subsets; each of these subsets is called an *orientation* of V.
  - For a basis $v_1, \ldots, v_n$, we write $[v_1, \ldots, v_n]$ for the orientation to which it belongs, and the other orientation is denoted $-[v_1, \ldots, v_n]$.
  - On $\mathbb{R}^n$, we define the *usual orientation* (or *standard orientation*) to be $[e_1, \ldots, e_n]$.
- Volume element
  - The characterization of $\det \in \Lambda^n(\mathbb{R}^n)$ by the property $\det(e_1, \ldots, e_n) = 1$ is not available in a general vector space V since there is no "standard basis".
  - But now suppose V has an inner product T, and consider orthonormal bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$.
  - If $w_i = \sum_j a_{ij} v_j$, then $A = (a_{ij})$ is an orthogonal matrix: $\delta_{ij} = T(w_i, w_j) = \sum_k a_{ik} a_{jk}$, or in other words $A^\top = A^{-1}$; hence $\det A = \pm 1$.
  - Thus, if $\omega(v_1, \ldots, v_n) = \pm 1$, then $\omega(w_1, \ldots, w_n) = \pm 1$.
  - If we moreover have an orientation μ, then there is a unique ω with $\omega(v_1, \ldots, v_n) = 1$ for an orthonormal oriented basis $[v_1, \ldots, v_n] = \mu$.
  - This is called the *volume element* of V determined by T and μ.
  - In $\mathbb{R}^n$, det is the volume element determined by the standard inner product and orientation.
  - The name comes from the fact that $\det(v_1, \ldots, v_n)$ is the volume of the parallelepiped spanned by $v_1, \ldots, v_n$.
- Tangent vectors
  - For fixed $p \in \mathbb{R}^n$, the set of all pairs (p, v) with $v \in \mathbb{R}^n$ is denoted $\mathbb{R}^n_p$ and is called the *tangent space to* $\mathbb{R}^n$ *at* p.
  - Of course, $\mathbb{R}^n_p$ is in bijection with $\mathbb{R}^n$ itself, and therefore is a vector space (and has a standard basis, standard inner product, standard orientation, etc.).
  - We write v_p for (p, v).
  - The *endpoint* of v_p is the point $p + v$.
- Vector fields
  - A *vector field* on $\mathbb{R}^n$ is a function F on $\mathbb{R}^n$ such that $F(p) \in \mathbb{R}^n_p$ for each p.
  - The *component functions* $F^1, \ldots, F^n \colon \mathbb{R}^n \to \mathbb{R}$ of F are given by $F(p) = F^1(p) \cdot (e_1)_p + \cdots + F^n(p) \cdot (e_n)_p$.
  - We say that F is a $\mathcal{C}^k$ vector field if each F^i is a $\mathcal{C}^k$ function.

  - Given vector fields F, G and a smooth function f, we can define vector fields $F + G$ and $f \cdot F$, and a function $\langle F, G \rangle$ pointwise: $(F + G)(p) = F(p) + G(p)$ and so on.
- Divergence and curl
  - The *divergence* $\text{div}F$ of a vector field is $\sum_{i=1}^{n} D_i F^i$.
  - If we consider the "vector of differential operators" $\nabla = \sum_{i=1}^{n} D_i \cdot e_i = (D_1, \dots, D_n)$, we can write $\text{div}F = \langle \nabla, F \rangle$.
  - Similarly, in $\mathbb{R}^3$, we can write $(\nabla \times F)$; this is called the *curl* of F.
  - The names "divergence" and "curl" comes from physics you may have seen; we will discuss it later.
- Differential forms
  - A *differential form of degree* k (or just k*-form*) on $\mathbb{R}^n$ is a function ω with $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$ for each $p \in \mathbb{R}^n$.
  - If $\varphi_1(p), \dots, \varphi_n(p)$ is dual basis to $(e_1)_p, \dots, (e_n)_p$, we can write

$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\dots,i_k}(p) \cdot [\varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p)]$$

  - We say that ω is a $\mathcal{C}^l$ k-form if each $\omega_{i_1,\dots,i_k}$ is $\mathcal{C}^l$. As usual, we only really care about the case $l = \infty$.
  - We can define the sum $\omega + \eta$, multiple $f \cdot \omega$, and $\omega \wedge \eta$ of forms pointwise: $(\omega + \eta)(p) = \omega(p) + \eta(p)$ and so on.
  - We also consider a function as a 0-form and write $f \wedge \omega$ for $f \cdot \omega$.
- Differential of a function
  - If $f \colon \mathbb{R}^n \to \mathbb{R}$, we have $\mathrm{D}f(p) \in \Lambda^1(\mathbb{R}^n)$; using the correspondence $\mathbb{R}^n_p \cong \mathbb{R}^n$, we thus obtain a 1-form $\mathrm{d}f$:

$$\mathrm{d}f(p)(v_p) = \mathrm{D}f(p)(v).$$

  - We write $x^i \colon \mathbb{R}^n \to \mathbb{R}$ for the function returning the i-th coordinate of a point.
  - Warning: there is potential for confusion: sometimes, we write $(x^1, \dots, x^n)$ for the coordinates of a *given* point, so each x^i is a number. Now, we are writing x^i for the coordinate *function*, so $x^i(a^1, \dots, a^n) = a^i$.
  - We now consider the 1-form $\mathrm{d}x^i$. We have

$$\mathrm{d}x^i(p)(v_p) = \mathrm{D}x^i(p)(v) = v^i.$$

  - Hence $\mathrm{d}x^1(p), \dots, \mathrm{d}x^n(p)$ is the dual basis to $(e_1)_p, \dots, (e_n)_p$, and we can write any k-form ω as

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1,\dots,i_k}(p) \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

  - Theorem 4-7: if $f \colon \mathbb{R}^n \to \mathbb{R}$ is differentiable, then

$$\mathrm{d}vf = \frac{\partial f}{\partial x^1} \cdot \mathrm{d}x^1 + \cdots + \frac{\partial f}{\partial x^n} \cdot \mathrm{d}x^n.$$

    The proof: $\mathrm{d}f(p)(v_p) = \mathrm{D}f(p)(v) = \sum_{i=1}^{n} v^i \cdot \frac{\partial f}{\partial x^i}(p) = \sum_{i=1}^{n} \mathrm{d}x^i(p)(v_p) \cdot \frac{\partial f}{\partial x^i}(p)$.
- Pullbacks of differential forms
  - Given a differentiable function $f \colon \mathbb{R}^n \to \mathbb{R}^m$, we have $\mathrm{D}f(p) \colon \mathbb{R}^n \to \mathbb{R}^m$; thus identifying $\mathbb{R}^n, \mathbb{R}^m$ with $\mathbb{R}^n_p, \mathbb{R}^m_p$, we obtain a linear map $f_* \colon \mathbb{R}^n_p \to \mathbb{R}^m_p$:

$$f_*(v_p) = \big(\mathrm{D}f(p)(v)\big)_{f(p)}$$

- We define the *pullback* of a k-form ω on $\mathbb{R}^m$ to be k-form $f^*\omega$ on $\mathbb{R}^n$ given by $(f^*\omega)(v_1, \ldots, v_k) = \omega\big(f(p)\big)(f_* v_1, \ldots, f_* v_k)$.
- Theorem 4-8:
  (1) In particular, $f^* \mathrm{d}x^i = \mathrm{d}f^i$ (and more generally: $f^*\mathrm{d}g = \mathrm{d}(g \circ f)$)
  (2) $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$
  (3) $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$
  (4) $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$
- Part (1) follows from the chain rule
- **Exercise:** Prove (2)-(4)

Exercises:

- Prove (2)-(4) of Theorem 4-8
- Spivak 4-14

4-14. Let c be a differentiable curve in $\mathbf{R}^n$, that is, a differentiable function c: $[0,1] \to \mathbf{R}^n$. Define the tangent vector v of c at t as $c_*((e_1)_t) = ((c^1)'(t), \ldots, (c^n)'(t))_{c(t)}$. If f: $\mathbf{R}^n \to \mathbf{R}^m$, show that the tangent vector to $f \circ c$ at t is $f_*(v)$.