Math 435: Lecture 30

March 29, 2024

Reference: Tapp, pp. 24-27

Topics:

- Curvature
  - We want to quantify how sharply a (regular) curve is turning.
  - As we mentioned, this should be measured by $\vec{a}^{\perp}$, adjusted to account for the effects of velocity.
  - One way of saying this is that the curvature function $\kappa(t)$ should be independent of the parametrization; i.e., if $\tilde{\gamma} = \gamma \circ \varphi$ is a reparametrization of γ, we should have that the curvature $\tilde{\kappa}$ of $\tilde{\gamma}$ satisfies $\tilde{\kappa} = \kappa \circ \varphi$.
  - Another desideratum is simply that as the radius of a circle grows, its curvature should decrease. Specifically, we will have that a circle of radius R have curvature $\kappa(t) = 1/R$.
  - To see how to fulfill the first condition, let $\tilde{\vec{a}}$ be the acceleration of $\tilde{\gamma}$, and let us compute $\tilde{\vec{a}}^{\perp}$.

    We have
    $$\tilde{\vec{v}}(u) = \tilde{\gamma}'(u) = \varphi'(u) \cdot \gamma'(\varphi(u)) = \varphi'(u) \cdot \vec{v}(\varphi(u))$$
    and
    $$\tilde{\vec{a}}(u) = \tilde{\gamma}''(u) = \varphi'(u)^2 \cdot \vec{a}(\varphi(u)) + \varphi''(u) \cdot \vec{v}(\varphi(u))$$
    and hence
    $$\tilde{\vec{a}}^{\perp}(u) = \varphi'(u)^2 \cdot \vec{a}^{\perp}(\varphi(u)).$$
    In conclusion (using our usual abuse of notation and writing $\vec{v}$ and $\vec{a}^{\perp}$ in place of $\vec{v} \circ \varphi$ and $\vec{a}^{\perp} \circ \varphi$):
    $$\tilde{\vec{v}} = (\varphi') \cdot \vec{v} \quad \text{and } \tilde{\vec{a}}^{\perp} = (\varphi')^2 \vec{a}^{\perp}.$$
  - So we see that the reparametrization scales $\vec{v}$ and $\vec{a}$ by a factor φ' and $(\varphi')^2$, respectively, so that the quantity $\frac{|\vec{a}^{\perp}|}{|\vec{v}|^2}$ is inavariant:
    $$\frac{|\vec{a}^{\perp}|}{|\vec{v}|^2} = \frac{|\tilde{\vec{a}}^{\perp}|}{|\tilde{\vec{v}}|^2}.$$
  - We therefore define the curvature of γ to be $\kappa(t) = \frac{|\vec{a}^{\perp}|}{|\vec{v}|^2}$.
  - We then have $|\vec{a}^{\perp}| = \kappa(t) \cdot |\vec{v}|^2$, in accordance with out intuition that the size of $\vec{a}^{\perp}$ should increase both with curvature and speed.
  - Note that for a circle $\gamma(t) = (R\cos(t), R\sin(t))$, we have $\vec{v} = (-R\sin t, R\cos t)$ and $\vec{a} = (-R\cos t, -R\sin t)$ (note that $\vec{v} \perp \vec{a}$ as it should be, since γ is unit speed; moreover, $\vec{a}$ points toward the origin, indicating that there is a "central force") and hence $\kappa(t) = |\vec{a}^{\perp}|/|\vec{v}|^2 = |\vec{a}|/|\vec{v}|^2 = R/R^2 = 1/R$, as desired.
  - Proposition 1.31: if γ is unit speed, then we simply have $\kappa(t) = |\vec{a}(t)|$ (since then $\vec{a} = \vec{a}^{\perp}$ and $|v| = 1$.
  - This gives another perspective on curvature. We can imagine the tangent vector $\vec{(t)}$ of a unit speed curve as tracing out a curve on the unit-sphere, indicating at each point in time in which direction the curve is moving.

  - Then the curvature tells us how fast this "direction-curve" is moving, or in other words how fast the direction of γ is changing.
- Unit tangent and normal
  - If γ is a regular curve, we define its *unit tangent* (at time t) vector to be $\vec{t}(t) := \vec{v}(t)/|\vec{t}|$.
  - The *unit normal* vector of γ is only defined whenever $\kappa(t) \neq 0$, and is given by $\vec{n}(t) = \vec{a}^{\perp}(t)/|\vec{a}^{\perp}(t)|$.
  - By construction, $\{\vec{t}, \vec{n}\}$ is orthonormal.
  - Proposition 1.34: if γ is regular, then $\kappa(t) = |\vec{t}\,'|/|\vec{v}|$.
    We have
    $$\vec{a} = \vec{v}\,' = (|\vec{v}|\vec{t})' = |\vec{v}|'\vec{t} + |v|\vec{t}\,'.$$
    Since $|\vec{t}|$ is constant, we have $\vec{t} \perp \vec{t}\,'$, and hence $\vec{a}^{\parallel} = |v|'\vec{t}$ and $\vec{a}^{\perp} = |v|\vec{t}\,'$.
    Hence $\kappa = |\vec{a}^{\perp}|/|\vec{v}|^2 = |\vec{t}|/|\vec{v}|$.
  - Thus, can say in general, that the rate of change of the direction "per unit of speed".
  - Also note that the proof showed that $\vec{a}^{\parallel} = |\vec{v}|'\vec{t}$, as we saw before: the component of $\vec{a}$ along $\vec{v}$ is the rate of change of speed.
  - Next, the above also gave $\vec{t}\,' \parallel \vec{a}^{\perp}$, whence
    $$\vec{n} = \frac{\vec{a}^{\perp}}{|\vec{a}^{\perp}|} = \frac{\vec{t}\,'}{|\vec{t}\,'|}.$$
    Thee both capture the idea that $\vec{n}$ points in the direction in which γ is bending.
  - Combining the above equations gives Proposition 1.35: if γ is regular, then whenever $\kappa \neq 0$, we have $\vec{t}\,' = \kappa|\vec{v}|\vec{n}$, and hence
    $$\kappa|v| = \langle \vec{t}\,', \vec{n} \rangle = -\langle \vec{n}\,', \vec{t} \rangle$$
    The last equation uses the general fact (Proposition 1.17 (2)) that if $F, G\colon I \to \mathbb{R}^n$ are orthogonal, then $\langle F', G \rangle = -\langle F, G' \rangle$, as follows immediately from the Leibniz rule for the inner product.