Math 435: Lecture 41

April 24, 2024

Reference: Tapp, pp. Chapters 4 and 5

Topics:

- Second fundamental form in local coordinates
  - As with the first fundamental form, we can also represent the second fundamental form in local coordinates.
  - That is, given a surface patch $\sigma\colon U \xrightarrow{\sim} V \subset S$, we can consider the function $(\mathcal{F}_2)_{\mathbf{q}}\colon \mathbb{R}^2 = \mathrm{T}_{\mathbf{q}}U \to \mathbb{R}$ given by $(\mathcal{F}_2)_{\mathbf{q}}(\vec{v}\,) = \mathrm{II}_{\mathbf{p}}(\mathrm{d}\sigma_{\mathbf{q}}(\vec{v}\,))$.
  - As before, we can write this in the form

$$\mathcal{F}_2 = e\,\mathrm{d}u^2 + 2f\,\mathrm{d}u\,\mathrm{d}v + g\,\mathrm{d}v^2$$

    where now $e = \mathrm{II}(\sigma_u)$, $f = \mathcal{W}(\sigma_u, \sigma_v)$, and $g = \mathrm{II}(\sigma_v)$.
  - We then obtain such formulas as

$$K = \frac{eg - f^2}{EG - F^2} \qquad H = \frac{eG - 2fF + gE}{EG - F^2} \qquad \{k_1, k_2\} = \{H - \sqrt{H^2 - K}, H + \sqrt{H^2 - K}\}.$$

- Total Gaussian curvature
  - Given a plane curve $\gamma\colon [a,b] \to \mathbb{R}$, its *total signed curvature* $\int_a^b \kappa_{\mathrm{s}}\,\mathrm{d}t$ can be interpreted as the *signed length* of the curve $\vec{t}\colon [a,b] \to \mathrm{S}^1$, where by "signed" length we mean $\int_{[a,b]} \vec{t}^{\,*}\,\mathrm{d}V$, where $\mathrm{d}V$ is the volume form on the circle (with its standard orientation).
  - Indeed, the oriented unit tangent vector at $\vec{v} \in \mathrm{S}^1$ is precisely $R_{90}\vec{v}$, and hence

$$\vec{t}^{\,*}\,\mathrm{d}V(\vec{e}_1) = \mathrm{d}V(\vec{t}^{\,\prime}) = \langle \vec{t}^{\,\prime}, R_{90}\vec{t}\,\rangle = \kappa_{\mathrm{s}} = \kappa_{\mathrm{s}}\,\mathrm{d}t(\vec{e}_1).$$

  - The *total signed curvature* $\int_S K\,\mathrm{d}V_S$ of a surface can be given a similar interpretation, namely as the *signed area* $\int_S \vec{N}^{\,*}\,\mathrm{d}V_{\mathrm{S}^2}$ of the Gauss map $\vec{N}\colon S \to \mathrm{S}^2$.
  - (In fact, this was Gauss' original definition of K: as the "infinitesimal signed area of the Gauss map".)
  - In other words, the claim is that $\vec{N}^{\,*}\,\mathrm{d}V_{\mathrm{S}^2} = K\,\mathrm{d}V_S$. To prove this, let $\vec{u}, \vec{v} \in \mathrm{T}_{\mathbf{p}}S$ be an orthonormal basis of tangent vector at some point so that $\mathrm{d}V_S(\vec{u}, \vec{v}\,) = 1$ by definition of $\mathrm{d}V$ and $\vec{u} \times \vec{v} = \vec{N}(\mathbf{p})$ by definition of $\vec{N}$.
  - Let us write $-\mathrm{d}\vec{N}_{\mathbf{p}}(\vec{u}\,) = a\vec{u} + b\vec{v}$ and $-\mathrm{d}\vec{N}_{\mathbf{p}}(\vec{v}\,) = c\vec{u} + d\vec{v}$. Then, by definition, $K(\mathbf{p}) = \det(\mathcal{W}_{\mathbf{p}}) = \det(-\mathrm{d}\vec{N}_{\mathbf{p}}) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.
  - On the other hand,

$$\begin{aligned}
(\vec{N}^{\,*}\,\mathrm{d}V_{\mathrm{S}^2})(\vec{u}, \vec{v}\,) &= \mathrm{d}V_{\mathrm{S}^2}(\mathrm{d}\vec{N}_{\mathbf{p}}(\vec{u}\,), \mathrm{d}\vec{N}_{\mathbf{p}}(\vec{v}\,)) \\
&= \langle\, \mathrm{d}\vec{N}_{\mathbf{p}}(\vec{u}\,) \times \mathrm{d}\vec{N}_{\mathbf{p}}(\vec{v}\,), \vec{N}(\mathbf{p})\rangle \\
&= \langle (a\vec{u} + b\vec{v}\,) \times (c\vec{u} + d\vec{v}\,), \vec{N}(\mathbf{p})\rangle \\
&= \langle (ad - bc)\vec{N}(\mathbf{p}), \vec{N}(\mathbf{p})\rangle \\
&= ad - bc \\
&= K(\mathbf{p})\,\mathrm{d}V_{\mathrm{S}}(\vec{u}, \vec{v}\,)
\end{aligned}$$

as desired.

- Normal sections
  - Recall that we defined the normal curvature $\mathrm{II}_{\mathbf{p}}(\vec{v})$ of an oriented surface S at a point $\mathbf{p}$ in the direction $\vec{v}$ to be the normal curvature $\langle \vec{N}(\mathbf{p}), \gamma''(t_0)\rangle$ of *any* unit-speed curve γ on S with $\gamma'(t_0) = \vec{v}$.
  - However, one can also define them in terms of certain canonically defined curves called *normal sections.*
  - We consider the plane P through $\mathbf{p}$ parallel to $\vec{N}(\mathbf{p})$ and $\vec{v}$, i.e., with normal vector $\vec{n} = \vec{N}(\mathbf{p}) \times \vec{v}$.
  - We now intersect P with S, and we claim that there is some small open set $U \subset \mathbb{R}^3$ containing $\mathbf{p}$ such that the intersection $S \cap P \cap U$ is a smooth curve, i.e., a smooth 1-manifold.
  - Indeed, P is the zero-set of the function $F(\vec{x}) = \langle \vec{x} - \mathbf{p}, \vec{n}\rangle$ with derivative $\mathrm{d}F(\mathbf{p}) = \vec{n}^{\top}$ (i.e., with gradient $\nabla F(\mathbf{p}) = \vec{n}$). And since S it is a 2-manifold, there is some neighbourhood W of $\mathbf{p}$ and smooth function $G\colon W \to \mathbb{R}$ with regular value 0 such that $W \cap S = G^{-1}(0)$; moreover, we have that $\nabla G(\mathbf{p}) \parallel \vec{N}(\mathbf{p})$, i.e., $\mathrm{d}G(\mathbf{p}) = \vec{u}^{\perp}$ for some $\vec{u} \parallel \vec{N}(\mathbf{p})$.
  - It follows that the function $H\colon U \to \mathbb{R}^2$ given by $H(\vec{x}) = (F(\vec{x}), G(\vec{x}))$ satisfies $H^{-1}(0) = S \cap P \cap W$, and $\mathrm{d}H(\mathbf{p})$ has linearly independent (in fact, orthogonal) rows $\vec{n}^{\top}$ and $\vec{u}^{\top}$, and hence has $\mathbf{p}$ as a regular point.
  - It then follows that 0 is a regular value of the restriction of H to some small neighbourhood $U \subset W$ of $\mathbf{p}$, and hence by the inverse function theorem, $C = H^{-1}(0) \cap U = P \cap S \cap U$ is a 1-manifold as claimed.
  - We can thus find a parametrization C near $\mathbf{p}$ by a unit-speed curve $\gamma\colon (\varepsilon, \varepsilon) \to \mathbb{R}^3$ with $\gamma(0) = \mathbf{p}$.
  - The smooth curve γ is called the *normal section* of S at $\mathbf{p}$.
  - By definition, it lies entirely inside the plane P, and hence γ'' lies inside P as well. Since $\gamma''(0) \perp \gamma'(0)$, it follows that $\gamma'' \parallel \vec{N}(\mathbf{p})$, and hence that
  $$\kappa(0) = |\gamma''(0)| = \langle \gamma''(0), \vec{N}(\mathbf{p})\rangle = |\mathrm{II}_{\mathbf{p}}(\vec{v})|.$$
  - We conclude that *the absolute value of the normal curvature* $\mathrm{II}_{\mathbf{p}}(\vec{v})$ *is the curvature at* $\mathbf{p}$ *of the normal section in the direction* $\vec{v}$.
- Geodesics
  - Let γ be a unit-speed curve on an oriented surface S with unit normal vector field $\vec{N}$.
  - We can decompose $\vec{a} = \gamma''$ as $\vec{a} = \vec{a}^{\parallel} + \vec{a}^{\perp}$ with respect to $\vec{N}$.
  - We have already identified the second term in terms of the normal curvature $\vec{a}^{\perp} = \kappa_{\mathrm{n}}\vec{N}$.
  - Since γ is unit-speed, $\vec{a}$ is orthogonal to both $\vec{v}$ and $\vec{N}$, and hence is parallel to $R_{90}\vec{v} := \vec{N} \times \vec{v}$, the "90 degree counter-clockwise rotation of γ' with respect to the chosen orientation".
  - We now set $\kappa_{\mathrm{g}} = \langle \gamma'', R_{90}\vec{v}\rangle$, so that we have $\vec{a}^{\parallel} = \kappa_{\mathrm{g}} R_{90}(\vec{v})$ and hence
  $$\vec{a} = \kappa_{\mathrm{n}} \cdot \vec{N} + \kappa_{\mathrm{g}} \cdot R_{90}(\vec{v}).$$
  - The quantity κ_{g} is called the *geodesic curvature* of γ, and measure the bending of γ *relative to* the surface.
  - If S is a plane, then κ_{g} is just the signed curvature.

- We see that $\kappa_g = 0$ if and only if $\gamma'' \parallel \vec{N}$; note that whether this holds is independent of orientation.
- Thus, in general, we define a *geodesic* on S to be a regular curve γ such that $\gamma''(t)$ is orthogonal to S for all t.
- Hence, if γ is unit-speed, it is a geodesic if and only if $\kappa_g = 0$.
- In general, we have: *every geodesic γ has constant speed.* This follows immediately from the fact that $\gamma' \perp \gamma''$.
- On a plane, the geodesics are precisely the lines with constant-speed parametrization.
- We now state two fundamental facts about geodesics without proof.
- Proposition 5.3: for each $\mathbf{p} \in S$ and each $\vec{v} \in \mathrm{T}_{\mathbf{p}}S$, there exists a geodesic $\gamma_{\vec{v}} : (a, b) \to S$ (where $0 \in (a, b)$) with $\gamma_{\vec{v}}(0) = \mathbf{p}$ and $\gamma'_{\vec{v}}(0) = \vec{v}$.
  Moreover, it is unique in the sense that for any other geodesic $\hat{\gamma}_{\vec{v}} : (\hat{a}, \hat{b}) \to S$ with these properties, $\gamma_{\vec{v}}$ and $\hat{\gamma}_{\vec{v}}$ agree on their common domain $(a, b) \cap (\hat{a}, \hat{b})$.
- Note that, while we can sometimes choose the domain (a, b) to be all of $\mathbb{R}$ (for example, when S is a plane), this isn't always the case.
  For example, if S is the puncture xy-plane $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0 \text{ and } (x, y) \neq (0, 0)\}$, and we take $\mathbf{p} = (-1, 0, 0)$ and $\vec{v} = (1, 0, 0)$, then the maximal domain of $\gamma_{\vec{v}}$ is $(-\infty, 1)$.
- This proposition is important because it allows us identify *all* the geodesics on a given surface.
- For example, *every geodesic on* S^2 *is (a part of) a great circle* $\gamma(t) = \cos(at)\vec{u} + \sin(at)\vec{v}$, where $\vec{u}, \vec{v} \in \mathrm{S}^2$ are orthogonal.
  Indeed, the great circles are clearly geodesic (since $\gamma'(t) = -a\sin(at)\vec{u} + a\cos(at)\vec{v} \perp \gamma(t) = \vec{N}(\gamma(t))$), hence for any $\mathbf{p} \in \mathrm{S}^2$ and $\vec{v} \in \mathrm{T}_{\mathbf{p}}S$, the unique geodesic $\gamma_{\vec{v}}$ through $\mathbf{p}$ with $\gamma'_{\vec{v}}(0) = \vec{v}$ must be the great circle $\gamma_{\vec{v}}(t) = \cos(at)\mathbf{p} + \sin(at)\vec{v}$.
- By a similar argument, we can see that *every geodesic on the cylinder* $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ *is a helix* $\gamma(t) = (\cos t, \sin t, ct)$.
- Next, Corollary 5.23: a curve γ is a geodesic if and only if it is *locally length-minimizing.* This means that for any t_0 in the domain of γ, there is some $\varepsilon > 0$ such that for any $t_1, t_2 \in (t_0 - \varepsilon, t_0 + \varepsilon)$, if we set $\mathbf{p} = \gamma(t_1)$ and $\mathbf{q} = \gamma(t_2)$, then $\gamma|_{[t_1,t_2]}$ is the shortest path on S from $\mathbf{p}$ to $\mathbf{q}$, i.e., any other curve $\alpha\colon [a, b] \to S$ from $\mathbf{p}$ to $\mathbf{q}$ on S has greater arc-length than $\gamma|_{[t_1,t_2]}$:

$$\int_a^b |\alpha'(t)|\,\mathrm{d}t \geq \int_{t_1}^{t_2} |\gamma(t)|\,\mathrm{d}t = |\gamma'|(t_2 - t_1)$$

  where for the last equation, we are using that γ has constant speed.
- In particular this tells us immediately that (Corollary 5.24): geodesics are intrinsic.
- This gives us a second proof that the helices are the geodesics on the cylinder, since we know what the geodesics in the plane are.

• Gauss-Bonnet
  - We finish our exploration of surfaces (though there is much more we are not covering – just look at some of the other sections in Tapp's book!) with the spectacular *Gauss-Bonnet* theorem (which we will discuss, but not prove).