Math 435 course outline

Spring 2024

Contents

Math 435: Lecture 1

January 8, 2024

Reference: Spivak, pp. 1-10

Topics:

- $\mathbb{R}^n$ (Spivak p. 1)
  - Vector space structure
  - Norm $\|x\|$ and inner product $\langle x, y\rangle$ and their basic properties (Spivak Theorems 1-1 and 1-2)
  - Linear maps $T\colon \mathbb{R}^n \to \mathbb{R}^m$, their representing matrices, composition.
- Subsets of Euclidean space
  - Closed and open rectangles in $\mathbb{R}^n$
  - Open and closed sets
  - Interior, exterior, and boundary points of a set
  - Open covers, compactness
  - Heine-Borel (compact if and only if closed and bounded – Spivak (1-3)-(1-7) and Problem 1-20)

Exercises:

- Spivak 1-7

1-7. A linear transformation T: $\mathbf{R}^n \to \mathbf{R}^n$ is **norm preserving** if $|T(x)| = |x|$, and **inner product preserving** if $\langle Tx, Ty\rangle = \langle x, y\rangle$.

(a) Prove that T is norm preserving if and only if T is inner-product preserving.

(b) Prove that such a linear transformation T is 1-1 and T^{-1} is of the same sort.

Math 435: Lecture 2

January 10, 2024

Reference: Spivak, pp. 7-33

Topics:

- Heine-Borel (compact if and only if closed and bounded – Spivak (1-3)-(1-7) and Problem 1-20)
  - Spivak does the "$\Leftarrow$" direction in in three steps:
  - (i) A closed finite interval is compact
    (This part makes essential use of the *completeness* of $\mathbb{R}$ – i.e., the *least upper bound property.*)
  - (ii) The product of compact sets is compact; hence a closed rectangle is compact
    (This is proven using the so-called "tube lemma", Spivak 1-4)
  - (iii) A closed subset of a compact set is compact
    (This part is easy!)
  - The "$\Rightarrow$" direction is easy and is left as an exercise (Problem 1-20)
- A function $f\colon A \to \mathbb{R}^m$ (with $A \subset \mathbb{R}^n$) is continuous if and only if the preimage of any open set is open (Spivak Theorem 1-8)
  - Here, "continuous" means that $\lim_{x\to a} f(x) = f(a)$ for all $a \in A$, or equivalently: for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \varepsilon$.
  - This is equivalent to each component of f being continuous (Spivak Problem 1-24).
- The continuous image of a compact set is compact (Spivak Theorem 1-9)
- Differentiation
  - Differentiability of functions $f\colon A \to \mathbb{R}^m$ in terms of linear approximation (Spivak Theorem 2-1 and the preceding discussion)
  - Derivatives in terms of component functions (Spivak Theorem 2-3 (3))
  - Jacobian matrix in terms of partial derivatives; continuously differentiable functions (Spivak Theorems 2-7 and 2-8)

Math 435: Lecture 3

January 12, 2024

Reference: Spivak, pp. 7-39

Topics:

- A little more on derivatives
  - Review of the three perspectives on the derivative: linear approximation, matrix of partial derivatives, directional derivative
  - $\mathcal{C}^\infty$ (smooth) functions and commutativity of partial derivatives (Spivak 2-5)
    Note: we will be mainly interested in smooth (hence continuously differentiable by Spivak Problem 2-1) functions.
  - Derivative at a maximum or minimum (Spivak 2-6)
  - Chain rule for derivatives (Spivak 2-2) and partial derivatives (Spivak 2-9)
- The all-important inverse function theorem (Spivak 2-11)
  - The easy direction:
    Exercise: suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and, for some $a \in \mathbb{R}^n$, there exists open subsets $V \subset \mathbb{R}^n$ containing $f(a)$ and $W \subset \mathbb{R}^n$ containing $f(a)$ such that $f\colon V \to W$ has a differentiable inverse $f^{-1}\colon W \to V$.
    Then $\det f'(a) \neq 0$, and in fact $(f^{-1})'(f(a)) = [f'(a)]^{-1}$.
  - Proof in the easy case $n = 1$.
  - Warning: inverse can still exist even if $\det f'(a) = 0$.
    Example: $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.

Math 435: Lecture 4

January 17, 2024

Reference: Spivak, pp. 40-43

Topics:

- Review of some properties of real numbers
  - Besides all of the usual properties of the real numbers (associativity, commutativity, distributivity, the properties of 1 and 0, and the basic properties of the ordering $a < b$), the real numbers have one more fundamental property:
  - The *least upper bound property*: any set of real numbers S which has an upper bound (i.e., a number M such that $x \leq M$ for all $x \in S$) has a *least upper bound* or *supremum* (i.e., an upper bound M such that $M \leq M'$ for any other upper bound M').
  - This implies the analogous "greatest lower bound" (or *infimum*) property. (**Exercise!**)
  - Some consequences:
  - (i) Any closed and bounded set of reals has a maximal element.
  - (ii) Any open interval of real numbers contains a rational number.
    (This uses three intermediate facts: (iii) the set of natural numbers is not bounded, (iv) for any positive real number, there is a smaller positive rational number, (v) if $b - a > 1$, then there is a rational number between a and b.)
  - The least upper bound property is also used in the proof other basic facts of calculus, such as the intermediate value theorem.
- Sample application of the inverse function theorem: the implicit function theorem (Spivak 2-12)
  - (The theorem is nice, but it is really the method of proof that is important, rather than theorem itself.)
  - The example of the circle
  - Proof in the case of functions $f\colon \mathbb{R}^2 \to \mathbb{R}$
  - Main idea: figure out how to turn f into a function $\mathbb{R}^2 \to \mathbb{R}^2$ with $\det f'(a,b) \neq 0$ so that you can apply the inverse function theorem.

Exercises:

(1) Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth and that the linear map $Df(a)$ is surjective for some $a \in \mathbb{R}^n$. Show that there is some open subset of $\mathbb{R}^m$ containing $f(a)$ which is in the image of f.

(2) Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth and that the linear map $Df(a)$ is injective for some $a \in \mathbb{R}^n$. Show that there is some open subset of $\mathbb{R}^n$ containing a on which f is injective.

Math 435: Lecture 5

January 19, 2024

Reference: Spivak, pp. 34-49

Topics:

- A few more remarks on the inverse function theorem:
  - The theorem still holds if $\mathcal{C}^1$ is replaced by $\mathcal{C}^k$ for any k (including $k = \infty$): i.e., if the original function f is $\mathcal{C}^k$, then the inverse to f guaranteed by the theorem is also $\mathcal{C}^k$ (see Spivak, Addendum 1)c
  - Some useful terminology: a map $\mathbb{R}^m \supset U \xrightarrow{f} V \subset \mathbb{R}^n$ is a $\mathcal{C}^k$*-diffeomorphism* if it is $\mathcal{C}^k$ and has a $\mathcal{C}^k$-inverse. (Spivak p. 109)
    f is a *local* $\mathcal{C}^k$*-diffeomorphism at* $a \in U$ if there is some open set $a \in U' \subset U$ such that $f(U')$ is open and the restriction $f\colon U' \to f(U')$ is a $\mathcal{C}^k$-diffeomorphism.
  - Thus the inverse function theorem says: if $\mathrm{D}f(a)$ is invertible, then f is a local diffeomorphism at a.
  - Now an example: let $U = \mathbb{R} \times (0,1) \subset \mathbb{R}^2$ and $V = \{\vec{x} \in \mathbb{R}^2 \mid 0 < |\vec{x}| < 1\}$.
    Let $f\colon U \to V$ be defined by $f(x,y) = (y\cos x, y\sin x)$.
    We have
    $$f'(x,y) = \begin{bmatrix} -y\sin x & \cos x \\ y\cos x & \sin x \end{bmatrix}$$
    and hence $\det f'(x,y) = -y(\sin^2 x + \cos^2 x) = -y \neq 0$.
    Hence by IFT, f is a local diffeomorphism at every point of U.
    However, f is clearly not a diffeomorphism, since it is not injective.
    Rather, $f\colon U \to f(U) = V - \{(y\cos t, y\sin y) \mid y \in (0,1)\}$ is a diffeomorphism with $U = (t, t+2\pi) \times (0,1)$ for any $t \in \mathbb{R}$.
- Definition of the integral of a bounded function defined on a closed rectangle $A \subset \mathbb{R}^n$
  - Partitions of closed rectangles into subrectangles (Spivak p. 46)
  - Volumes of rectangles $v(R)$, and lower and upper sums $L(f,P) = \sum_S m_S(f) \cdot v(S)$ and $U(f,P) = \sum_S M_S(f) \cdot v(S)$ (p. 47)
  - If P' refines P, then $L(f,P) \leq L(F,P')$ and $U(F,P') \leq U(f,P)$. (Spivak 3-1)
  - If P and P' are any two partitions, then $L(f,P') \leq U(f,P)$ (Spivak 3-2)
  - $f\colon A \to \mathbb{R}$ is integrable if it is bounded and $\sup_P L(f,P) = \inf_P U(f,P)$.
    This number is called the *integral* of f over A, denoted $\int_A f$ or $\int_A f(x_1, \ldots, x^n)\,\mathrm{d}x^1 \cdots \mathrm{d}x^n$ (or $\int_a^b f$ in the case $n = 1$). (Spivak p. 48)
  - A bounded f is integrable if and only if for every $\varepsilon > 0$ there is a partition P with $U(f,P) - L(f,P)$ (Spivak 3-3)
  - Examples: (i) constant function and (ii) $f\colon [0,1] \times [0,1] \to \mathbb{R}$ defined by $f(x,y) = 0$ if $x \in \mathbb{Q}$ and $f(x,y) = 1$ else

Exercises:

(1) Spivak 3-1

Problems. **3-1.** Let $f\colon [0,1] \times [0,1] \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Show that f is integrable and $\int_{[0,1]\times[0,1]} f = \frac{1}{2}$.

(2) Spivak 3-2

3-2. Let $f\colon A \to \mathbf{R}$ be integrable and let $g = f$ except at finitely many points. Show that g is integrable and $\int_A f = \int_A g$.

Math 435: Lecture 6

January 22, 2024

Reference: Spivak, pp. 50-51

Topics:

- Basic facts about integrals (Spivak, problems on p. 49 and Problem 3-14):
  - If f and g are integrable, then so is $f+g$, and $\int_A f + g = \int_A f + \int_A g$
  - If f is integrable, then so is $c \cdot f$, and $\int_A c \cdot f = c \cdot \int_A f$
  - If f and g are both integrable and $f \leq g$, then $\int_A f \leq \int_A g$
  - If f is integrable, then so is $f_+ = \max(0, f)$ and $f_- = \min(0, f)$
  - If f is integrable, then so is $|f|$, and $|\int_A f| = \int_A |f|$
  - If f and g are integrable, then so is $f \cdot g$.
  - The fundamental theorem of calculus: if f is continuous, then $F(x) = \int_a^x f$ is differentiable and $F' = f$.
- Measure zero
  - A set $A \subset \mathbb{R}^n$ has *measure 0* if for every $\varepsilon > 0$, there is a cover $\{U_1, U_2, \ldots,\}$ by closed (or equivalently, open) rectangles with $\sum_i v(U_i) < \varepsilon$.
  - A subset of a set with measure 0 has measure 0
  - Finite sets have measure 0.
  - Countable sets have measure 0 (take $v(U_i) < \varepsilon/2^i$)
  - $\mathbb{Q}$ has measure 0 (zigzag trick)
  - Theorem 3-4: A countable union of measure 0 sets has measure 0 (combine previous two tricks)

Exercise: Prove the above basic properties of integrals.

Math 435: Lecture 7

January 24, 2024

Reference: Spivak, pp. 51-56

Topics:

- *Content 0*: same thing as measure 0 but with finite covers.
  - Theorem 3-6: Compact and content 0 implies measure 0
    (Use open rectangles!)
  - Theorem 3-5: $[0,1]$ does not have content 0 (hence does not have measure 0 since it is compact); in fact, any finite cover has $\sum_i U_i \geq b-a$
  - However, $[0,1] \cap \mathbb{Q}$ does not have content 0 (since a finite union of closed intervals is closed, and $[0,1] \cap \mathbb{Q}$ has closure $[0,1]$).
  - The above basic facts about integrals
- Integrability and continuity
  - Theorem 3-8: A bounded function $f\colon A \to \mathbb{R}$ on a closed rectangle is integrable if and only if the set $B = \{x \mid f \text{ is not continuous at } x\}$ has measure zero.
  - Proof of the simple special case $B = \emptyset$ and under the assumption that f is *uniformly continuous*, meaning for all $\epsilon > 0$ there is a $\delta > 0$ such that $|\vec{x} - \vec{y}| < \delta(\varepsilon)$ implies $|f(\vec{x}) - f(\vec{y})| < \epsilon$ for any $x, y \in A$ (in fact, this is automatically true for any continuous function on a compact set):
    Fix $\epsilon > 0$. Then for any partition P such that each sub-rectangle has diameter $< \delta(\varepsilon/v(A))$, we have $U(f,P) - L(f,P) < (\varepsilon/v(A)) \cdot v(A) = \varepsilon$.
- Integration over more general bounded domains
  - The characteristic function $\chi_C(x)$ for $C \subset \mathbb{R}^n$ which is 1 for $x \in C$ and 0 else.
  - For $C \subset \mathbb{R}^n$ a bounded domain (so $C \subset A$ for some rectangle A) and a bounded function $f\colon C \to \mathbb{R}$, we define $\int_C f = \int_A f \cdot \chi_C$ (provided $f \cdot \chi_C$ is integrable – hence for example if both f and χ_C are integrable).

Exercises:

(1) Spivak 3-9

3-9. (a) Show that an unbounded set cannot have content 0.
(b) Give an example of a closed set of measure 0 which does not have content 0.

(2) Spivak 3-8

Problems. **3-8.** Prove that $[a_1,b_1] \times \cdots \times [a_n,b_n]$ does not have content 0 if $a_i < b_i$ for each i.

Math 435: Lecture 8

January 26, 2024

Reference: Spivak, pp. 56-62

Topics:

- Jordan-measurable sets
  - Theorem 3-9: the function χ_C is integrable if and only if the boundary of C has measure 0 (hence content 0, since a boundary is always closed, and C is bounded). (The reason is that the points of discontinuity of C are exactly the boundary points.)
  - Such a set C is called *Jordan-measurable* and $\int_C 1$ is its (n-dimensional) *content* or (n-dimensional) *volume* (or, in 1 and 2 dimensions, *length* or *area*).

  (Warning: not every open set is Jordan-measurable; later, we will introduce a generalized version of integration which is defined for bounded functions on any open set.)
- Fubini's theorem (3-10): if $f\colon A \times B \to \mathbb{R}$ is integrable, and $g_x\colon B \to \mathbb{R}$ defined by $g_x(y) = f(x,y)$ is integrable for all $x \in A$ and we define $G\colon A \to \mathbb{R}$ by $G(x) = \int_B g_x$, then we have $\int_{A\times B} f = \int_A G = \int_A \left(\int_B f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$.
  - Spivak proves a slightly stronger version of this theorem.
  - The hypothesis of the theorem holds whenever f is continuous.
  - Since integrability is not affected by changing the value of a function at finitely many points, the theorem still works if g_x is integrable for all but finitely many $x \in A$.
  - Of course, the same proof gives $\int_{A\times B} f = \int_B \left(\int_A f(x,y)\,\mathrm{d}x\right)\mathrm{d}y$.
  - Applying the theorem repeatedly, we have for $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $f\colon A \to \mathbb{R}$ sufficiently nice (e.g., continuous) that $\int_A f = \int_{a_n}^{b_n} \left(\cdots \left(\int_{a_1}^{b_1} f(x^1, \ldots, x^n)\,\mathrm{d}x^1\right)\cdots\right)\mathrm{d}x^n$. Thus, in this case, multivariable integrals are reduced to single-variable ones.
  - The proof:
    Any partition P_A of A and P_B of B gives a partition $P_A \times P_B$ of $A \times B$ with sub-rectangles $S_A \times S_B$ where S_A and S_B are sub-rectangles of P_A and P_B.
    For any fixed S_A and $x \in S_A$, we have $m_{S_A\times S_B} \le m_{S_B}(g_x)$ and hence
    $\sum_{S_B} m_{S_A\times S_B}(f)v(S_B) \le \sum_{S_B} m_{S_B}(g_x)v(S_B) = L(g_x, P_B) \le \int_B g_x = G(x)$ and hence
    $\sum_{S_B} m_{S_A\times S_B}(f)v(s_B) \le m_{S_A}(G)$. Hence:
    $L(f, P_A \times P_B) = \sum_{S_A,S_B}(f)v(s_A)v(s_B) \le \sum_{S_A} m_{S_A}(G)v(S_A) = L(G, P_A) \le \int_A G$.
    The same argument gives $\int_A G \le U(f, P_A \times P_B)$, hence we have $L(f, P_A \times P_B) \le \int_A G \le U(f, P_A \times P_B)$ and it follows that $\int_{A\times B} f = \int_A G$.

Exercises:

(1) Spivak 3-15

> **3-15.** Show that if C has content 0, then $C \subset A$ for some closed rectangle A and C is Jordan-measurable and $\int_A \chi_C = 0$.

(2) Spivak 3-26

3-26. Let $f\colon [a,b] \to \mathbf{R}$ be integrable and non-negative and let $A_f = \{(x,y)\colon a \le x \le b \text{ and } 0 \le y \le f(x)\}$. Show that A_f is Jordan-measurable and has area $\int_a^b f$.

Math 435: Lecture 9

January 29, 2024

Reference: Spivak, pp. 63-64

Topics:

- Partitions of unity
  - For a set $A \subset \mathbb{R}^n$ and an open cover $\mathcal{O}$ of A, a $\mathcal{C}^\infty$ *partition of unity for* A is a collection Φ of smooth functions φ defined on an open neighbourhood of A such that
    (1) $0 \leq \varphi(x) \leq 1$ for $x \in A$
    (2) each $x \in A$ has an open neighbourhood V such that all but finitely φ are 0 on V
    (3) $\sum_{\varphi \in \Phi} \varphi(x) = 1$ for all $x \in A$.
    We say Φ is *subordinate* to $\mathcal{O}$ if moreover
    (4) for each $\varphi \in \Phi$ there is some $U \in \mathcal{O}$ such that $\varphi = 0$ outside some closed subset of U. (One says that φ is *supported* on U.)
  - Theorem 3-11: for any A and $\mathcal{O}$, there exists partition of unity for A subordinate to $\mathcal{O}$.
  - The main ingredient in the proof is the existence of *smooth bump functions*: for any open $U \subset \mathbb{R}^n$ and any compact $K \subset U$, there is a smooth function f supported in U with $f = 1$ on K.
    This is constructed in Spivak Problem 2-26.
    The starting point is $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$.
    Then $g(x) = f(x)/(f(x) + f(1-x))$ is zero for $x \leq 0$ and 1 for $x \geq 1$.
  - The proof starts with a couple of reductions: first, any partition of unity for an open neighbourhood of A is also a partition of unity for A; hence, we may assume A is open. Next, we reduce to the case where A is compact, using that, for any open set U, there is a sequence of compact sets $K_1 \subset K_2 \subset \cdots$ with $\bigcup_i K_i = U$ and each K_i is contained in the interior of K_{i+1}.
  - Here is the proof when A is compact (using the existence of bump functions):
    For each $x \in A$, choose an open set $U_x \in \mathcal{O}$, and an open set V_x and compact set K_x with $x \in V_x \subset K_x \subset U_x$.
    Since A is compact, it is covered by finitely many of the V_x, say $V_{x_1}, \ldots, V_{x_N}$.
    Let ψ_i be a smooth bump function supported on U_{x_i} and equal to 1 on K_i.
    Then $\psi_1 + \cdots + \psi_N > 0$ on the open set $U = V_1 \cup \cdots \cup V_N$ containing A.
    Now define φ_i on U by letting $\varphi_i = \psi_i/(\psi_1 + \cdots \psi_N)$; note that φ_i may not be defined on all of U_i since the denominator can vanish somewhere on U_i.
    Hence, finally, let f be a bump function supported on U and equal to 1 on A. Then each $f\varphi_i$ is defined on all of U_i, and $\Phi = \{f \cdot \varphi_1, \ldots, f \cdot \varphi_N\}$ is the desired partition of unity.

Math 435: Lecture 10

January 31, 2024

Reference: Spivak, pp. 65-66

Topics:

- First application of partition of unity: integration on arbitrary open sets
  - Our current definition of $\int_A f$ for arbitrary A only works if A is bounded and bd$A$ has measure 0.
  - We would like a definition that works for arbitrary open sets A; but even a bounded open set may not be non-measure-0 boundary.
    (Example: Problem 3-11)
    We would also like to be able to integrate unbounded functions.
  - First, a reminder: a series $\sum_{i=1}^{\infty} a_i$ *converges absolutely* if $\sum_{i=1}^{\infty}|a_i|$ converges.
  - In this case, every reordering $\sum_{i=1}^{\infty} a_{k_i}$ of the series converges and has the same value.
  - (Moreover, the *Riemann series theorem* says that if $\sum_{i=1}^{\infty} a_i$ does *not* converge absolutely, then any $S \in \mathbb{R}$, there is a reordering a_{k_i} such that $\sum_{i=1}^{\infty} a_{k_i} = S$.)
  - Let A be open, let $\mathcal{O}$ be an open cover of A with $U \subset A$ for each $U \in \mathcal{O}$ and let Φ be a partition of unity subordinate to $\mathcal{O}$.
    Suppose $f\colon A \to \mathbb{R}$ is continuous outside of a set of measure 0, so that each $\int_A \varphi \cdot |f|$ exists.
  - We say that f is *integrable in the extended sense* if the series $\sum_{\varphi\in\Phi} \varphi \cdot \int|f|$ converges, and hence $\Sigma_{\varphi\in\Phi} \int_A \varphi \cdot f$ converges absolutely, and we define $\int_A f$ to be its sum.
  - (Though the convergence of $\sum_{\varphi\in\Phi} \varphi \cdot \int|f|$ is more than is needed to guarantee the absolute convergence of $\Sigma_{\varphi\in\Phi} \int_A \varphi\cdot f$, it *is* needed to ensure that this value is independent of the chosen partition of unity Φ, as is shown in Problem 3-38.)
  - Theorem 3-12:
    (1) $\int_A f$ is independent of the cover $\mathcal{O}$ and the partition of unity Φ.
    (2) If A and f are bounded (and A is open and f is continuous outside a set of measure 0), then $\int_A f$ exists.
    (3) This agrees with the old definition of $\int_A f$ when they are both defined.

Exercises:

(1) Spivak 3-36

3-36. (Cavalieri's principle). Let A and B be Jordan-measurable subsets of $\mathbf{R}^3$. Let $A_c = \{(x,y)\colon (x,y,c) \in A\}$ and define B_c similarly. Suppose each A_c and B_c are Jordan-measurable and have the same area. Show that A and B have the same volume.

(2) Spivak 3-38

3-38. Let A_n be a closed set contained in $(n, n+1)$. Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ satisfies $\int_{A_n} f = (-1)^n/n$ and $f = 0$ for $x \notin$ any A_n. Find two partitions of unity Φ and Ψ such that $\Sigma_{\varphi\in\Phi}\int_{\mathbf{R}} \varphi \cdot f$ and $\Sigma_{\psi\in\Psi}\int_{\mathbf{R}} \psi \cdot f$ converge absolutely to different values.

Math 435: Lecture 11

February 2, 2024

Reference: Spivak, pp. 66-76

Topics:

- Change of variables
  - Change of variables in one dimension: $\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$ ("set $u = g(x)$, then $\mathrm{d}u = g'(x)\,\mathrm{d}x$").
    (Suppose f is continuous and g is continuously differentiable.)
  - More concisely: $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g'$.
  - Proof: if $F' = f$, then $(F \circ g)' = (f \circ g) \cdot g'$, so the left side $F(g(b)) - F(g(a))$ and the right side is $(F \circ g)(b) - (F \circ g)(a)$.
  - Assuming $g\colon (a,b) \to (g(a), g(b))$ is a diffeomorphism, this can be written as $\int_{g((a,b))} f = \int_{(a,b)} (f \circ g) \cdot |g'|$.
  - In higher dimensions, we have:
    Theorem 3-13: if $A \subset \mathbb{R}^n$ is open and $g\colon A \to g(A) \subset \mathbb{R}^n$ is a diffeomorphism, and $f\colon g(A) \to \mathbb{R}$ is integrable, then $\int_{g(A)} f = \int_A (f \circ g)|\det g'|$.

  - Example: integrating in polar coordinates
  - The proof has several steps.
  - One first uses partitions of unity to reduce to the simple case is in which (i) A is a rectangle, and (ii) f is the constant function 1.
  - This case is proven by induction on n, the base case $n = 1$ being already known.
    For the induction step when $n > 1$, there is a further reduction using the inverse function theorem to the case in which (iii) $g^n(x) = x^n$, hence $g(x) = (g^1(x), \ldots, g^{n-1}(x), x^n)$.
    Now the claim follows from Fubini's theorem. Write $A = B \times [a_n, b_n]$ and define $g_{x^n}(x^1, \ldots, x^{n-1}) = (g^1(x^1, \ldots, x^n), \ldots, g^{n-1}(x^1, \ldots, x^n))$ for each $x^n \in [a, b]$, we have by induction that $\int_{g(B)} 1 = \int_B |\det g'_{x^n}|$.
    But we also have $\det g'_{x^n}(x^1, \ldots, x^{n-1}) = \det g'(x^1, \ldots, x^n)$ and hence, by Fubini:

$$\begin{aligned}
\int_{g(A)} 1 &= \int_{[a_n,b_n]} \int_{g(B \times x^n)} 1 \\
&= \int_{[a_n,b_n]} \Big(\int_{g_{x^n}(B)} 1 \Big)\,\mathrm{d}x^n \\
&= \int_{[a_n,b_n]} \Big(\int_B |\det g'_{x^n}| \Big)\,\mathrm{d}x^n \\
&= \int_{[a_n,b_n]} \Big(\int_B |\det g'(x^1, \ldots, x^n)|\,\mathrm{d}x^1 \ldots \mathrm{d}x^n \Big)\,\mathrm{d}x^n \\
&= \int_A |\det g'|.
\end{aligned}$$

  - It remains to explain how the reductions (i), (ii), (iii) are carried out.

- The reduction (i) involves a computation which looks like
  $$\int_{g(A)} f = \sum_{\varphi\in\Phi}\int_{g(A)} \varphi\cdot f = \sum_{\varphi\in\Phi}\int_A (\varphi\circ g)\cdot(f\circ g)\cdot|\det g'| = \int_A (f\circ g)\cdot|\det g'|$$
  where Φ is a partition of unity (which we can take to be subordinate to a cover by rectangles).
  In fact, this shows that it even suffices to prove the theorem in an *arbitrarily small* rectangle around each point in A.
- The reduction (ii) is a direct computation from the definition of the integral using, roughly speaking, that that $\sum_S m_S(f)\chi_S \le f \le \sum_S M_S(f)\chi(S)$ for any partition P.
- For the reduction (iii), an easy computation shows that, if the theorem holds for a given diffeomorphism $g_1\colon A\to g_1(A)$ and for a second diffeomorphism $g_2\colon g_1(A)\to g_2(g_1(A))$, then it is also true for $g_2\circ g_1\colon A\to g_2(g_1(A))$.
  Now fix $g\colon A\to g(A)$, and $a\in A$.
  If T is the linear transformation $Dg(a)$, then $(T^{-1}\circ g)'(a) = I$ by the chain rule, and it suffices to prove the claim for $T^{-1}\circ g$ since $g = T\circ(T^{-1}\circ g)$, and since we already proved the theorem for linear transformations.
  In other words, we may assume $g'(a) = I$.
  Now define $h\colon A\to\mathbb{R}^n$ by $h(x) = (g^1(x),\ldots,g^{n-1}(x),x^n)$. Then $\det h'(a) = \det I \ne 0$, hence $h\colon U\to h(U)$ is a diffeomorphism on some open neighbourhood U around a. Defining $k\colon h(U)\to\mathbb{R}^n$ by $k(y) = (y^1,\ldots,y^{n-1},g^n(h^{-1}(y)))$, we then have $g = k\circ h$, and hence it suffices to prove the claim for h and k, which are both of the desired form.

- Multilinear maps/tensors
  - A multilinear map on a vector space V (over $\mathbb{R}$) is a function $T\colon V^k\to\mathbb{R}$ which is linear in the i-th input whenever all the other inputs are held constant (for any $i = 1,\ldots,k$). In other words, given any $v_j\in V$ for all $j\ne i$, the function $V\to\mathbb{R}$ given by $v_i\mapsto T(v_1,\ldots,v_n)$ is linear.
  - Spivak also calls a multilinear map $V^k\to\mathbb{R}$ a *k-tensor*. (Note: it would be more usual to call it a *$(k,0)$-tensor*, but this abbreviation is reasonable since we will not be dealing with (k,l)-tensors for any $l>0$.)
  - $\mathcal{T}^k(V)$ is the set of all tensors over V; it is a vector space under pointwise addition and scalar multiplication.
  - There is an operation $\otimes\colon \mathcal{T}^k(V)\times\mathcal{T}^l(V)\to\mathcal{T}^{k+l}(V)$ defined by $(S\otimes T)(v_1,\ldots,v_{k+l}) = S(v_1,\ldots,v_k)\cdot T(v_{k+1},\ldots,v_{k+l})$.
  - **Exercise:** The operation $\otimes$ is bilinear and associative.
  - By associativity, we may as usual omit parentheses and write $S_1\otimes\cdots\otimes S_m$.

Math 435: Lecture 12

February 5, 2024

Reference: Spivak, pp. 76-77

Topics:

- Some example of vector spaces
  - Main example: linear subspaces of $\mathbb{R}^n$
  - The set of all $\mathcal{C}^k$ functions on a set $U \subset \mathbb{R}^n$ (this is infinite-dimensional!)
  - The set of all polynomials in n variables (also infinite-dimensional)
  - The set of all degree d polynomials in n variables (finite-dimensional)
  - Example from last time: the set $\mathcal{T}^k V$ of all k-tensors on V, i.e., multilinear maps $V^k \to \mathbb{R}$
- The axioms for a vector space:
  (i) associativity and commutativity of addition
  (ii) existence of the zero vector
  (iii) existence of additive inverses
  (iv) associativity of scalar multiplication
  (v) distributivity of scalar multiplication
  (vi) $1 \cdot v = v$
- From last time: **Exercise:** The operation $\otimes$ is bilinear and associative.
- By associativity, we may as usual omit parentheses and write $S_1 \otimes \cdots \otimes S_r$.
- The dual space and tensors
  - Note: $\mathcal{T}^1(V)$ is just the dual space V^*.
  - Recollection of dual bases: if V is a (finite-dimensional) vector space over $\mathbb{R}$, then for any basis $v_1, \ldots, v_n$ of V, there is a unique basis $\varphi_1, \ldots, \varphi_n$ of V^* with $\varphi_i(v_j) = \delta_{ij}$ called the *dual basis*.
    (And conversely, any basis of V^* is dual to a unique basis of V.)
    (δ_{ij} is the *Kronecker delta symbol*, which is defined to be 1 if $i = j$ and 0 if $i \neq j$.)
  - Theorem 4-1: if $v_1, \ldots, v_n$ is a basis for V with dual basis $\varphi_1, \ldots, \varphi_n$ of V^*, then the set of all k-fold tensor products $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ (with $i_1, \ldots, i_k \in \{1, \ldots, n\}$) is a basis for $\mathcal{T}^k(V)$, which therefore has dimension n^k.
  - The proof:
    First observation: $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(v_{j_1}, \ldots, v_{j_k}) = \delta_{i_1,j_1} \cdots \delta_{i_k,j_k}$.
    Second observation: $T = \sum_{i_1,\ldots,i_k=1}^n T(v_{i_1}, \ldots, v_{i_k})\, \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$
    (Applying both sides to an arbitrary input $(w_1, \ldots, w_k)$ and expressing each w_i as $w_i = \sum_{j=1}^n a_{i,j} v_j$, we get $\sum_{j_1,\ldots,j_k=1}^n a_{1,j_1}, \ldots, a_{k,j_k} T(v_{j_1}, \ldots, v_{j_k})$).
    Third observation: if $\sum_{i_1,\ldots,i_k=1}^n a_{i_1,\ldots,i_k}\, \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} = 0$, then by applying this to $v_{i_1}, \ldots v_{i_k}$, we obtain $a_{i_1,\ldots,i_k} = 0$.
  - Given a linear map $f \colon V \to W$, there is an induced map $f^* \colon \mathcal{T}^k(W) \to \mathcal{T}^k(V)$ given by $f^*T(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k))$.

Exercises:

- Check that $\mathcal{T}^k(V)$ satisfies the axioms of a vector space.
- The operation $\otimes$ is bilinear and associative.
- $f^*(S \otimes T) = f^*S \otimes f^*T$

Math 435: Lecture 13

February 7, 2024

Reference: Spivak, pp. 77-79

Topics:

- Inner products are examples of 2-tensors
  - An inner-product on a vector space V is a symmetric, positive-definite bilinear map $T \in \mathcal{T}^2(V)$.
  - Any (finite-dimensional) inner product space has an orthonormal basis (by the Gram-Schmidt process), and is thus isomorphic to $\mathbb{R}^n$ with the standard inner product.
- Alternating k-tensors
  - A k-tensor ω is *alternating* if switching two of the arguments in $\omega(v_1, \dots, v_k)$ changes the sign of the result.
  - Example: $\det \in \mathcal{T}^n(\mathbb{R}^n)$; it is the *unique* alternating n-tensor with $\det(e_1, \dots, e_n) = 1$.
  - We write $\Lambda^k(V) \subset \mathcal{T}^k(V)$ for the subspace consisting of alternating k-tensors.
  - Recall the sign $\operatorname{sgn} \sigma$ of a permutation $\sigma \in S_k$ of $\{1, \dots, k\}$, which is $(-1)^N$ where N is the number of pairs $1 \leq i < j \leq k$ with $\sigma(j) < \sigma(i)$ (or equivalently, the number of transpositions performed to produce σ).
  - For $T \in \mathcal{T}^k(V)$, we define $\operatorname{Alt}(T)$ by $\operatorname{Alt}(T)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in \mathrm{S}_k} \operatorname{sgn} \sigma \cdot T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$. Here, S_{k+l} is the set of permutations of $\{1, \dots, k+l\}$, i.e., bijections $\{1, \dots, k+l\} \to \{1, \dots, k+l\}$.
  - (Compare: $\det A = \sum_{\sigma \in \mathrm{S}_k} \operatorname{sgn} \sigma \cdot A_{1,\sigma(1)}, \dots, A_{k,\sigma(k)}$.)
  - Theorem 4-3: (1) $\operatorname{Alt}(T) \in \Lambda^k(V)$, (2) if $\omega \in \Lambda^k(V)$, then $\operatorname{Alt}(\omega) = \omega$, (3) $\operatorname{Alt}(\operatorname{Alt}(T)) = \operatorname{Alt}(T)$.
- The wedge product
  - $\wedge \colon \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is defined by $\omega \wedge \eta = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta)$.
  - The strange coefficient makes various formulas work out nicer later on.
    For now, note that for $\omega, \eta \in \Lambda^1(V) = \mathcal{T}^1(V) = V^*$, we have $(\omega \wedge \eta)(u, v) = \omega(u)\eta(v) - \eta(u)\omega(v)$.

Math 435: Lecture 14

February 9, 2024

Reference: Spivak, pp. 79-84

Topics:

- The wedge product
  - $\wedge\colon \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is defined by $\omega \wedge \eta = \frac{k!l!}{(k+l)!}\mathrm{Alt}(\omega \otimes \eta)$.
  - The strange coefficient makes various formulas work out nicer later on.
    For now, note that for $\omega, \eta \in \Lambda^1(V) = \mathcal{T}^1(V) = V^*$, we have $(\omega \wedge \eta)(u, v) = \omega(u)\eta(v) - \eta(u)\omega(v)$.
  - **Exercise:** Some basic facts about the wedge product:
    (i) $\wedge$ is bilinear
    (ii) $\wedge$ is "graded-commutative": $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$
    (iii) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$
  - $\wedge$ is also associative; in fact, we have (Theorem 4-4): $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta)$
  - The proof is more difficult then the above facts; it requires two lemmas:
  - Lemma 1: If $S \in \mathcal{T}^k(V)$ and $T \in \mathcal{T}^l(V)$ and $\mathrm{Alt}(S) = 0$, then $\mathrm{Alt}(S \otimes T) = \mathrm{Alt}(T \otimes S) = 0$.
  - Lemma 2: $\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \theta) = \mathrm{Alt}(\omega \otimes \eta \otimes \theta) = \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta))$.
  - The theorem follows from Lemma 2 which follows from Lemma 1
  - Proof of Lemma 1 when $k = 2$:
    Suppose $S \in \mathcal{T}^2(V)$ and $0 = \mathrm{Alt}(S)(v, w) = \frac{1}{2}(S(v, w) - S(w, v))$ for all v, w.
    We have $\mathrm{Alt}(S \otimes T)(v_1, \ldots, v_{l+2}) = \sum_\sigma \operatorname{sgn}\sigma \cdot S(v_{\sigma 1}, v_{\sigma 2})T(v_{\sigma 3}, \ldots, v_{\sigma(l+2)})$.
    But now for every σ there is an accompanying permutation $\sigma' = \sigma(12)$ with the opposite sign; since $S(v, w) = S(w, v)$, the sum breaks up into two parts which are the same but with opposite sign.
  - Again, by associativity, we may as usual omit parentheses and write $\omega_1 \wedge \cdots \wedge \omega_r$.
- The dimension of $\Lambda^k(V)$.
  - Theorem 4-5: if $v_1, \ldots, v_n$ is a basis for V with dual basis $\varphi_1, \ldots, \varphi_n$ of V^*, then the set of all k-fold wedge products $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$ is a basis for $\Lambda^k(V)$, which therefore has dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
  - Proof: given $\omega \in \Lambda^k(V) \subset \mathcal{T}^k(V)$, write $\omega = \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k}\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$.
    Thus $\omega = \mathrm{Alt}(\omega) = \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k}\mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$.
    But now $\mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$ is some multiple of $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$, which is zero if any two i_j are equal, and otherwise is ± 1 times $\varphi_{i'_1} \wedge \cdots \wedge \varphi_{i'_k}$ where $i'_1 < \ldots < i'_k$.
    Thus our putative basis is a spanning set.
    The proof of independence is as above, using that $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_{j_1}, \ldots v_{j_k}) = \delta_{i_1 j_1} \cdots \varphi_{i_k j_k}$ if $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$:
    If $0 = \omega = \sum_{i_1 < \ldots < i_k} a_{i_1,\ldots,i_k}\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ then $0 = \omega(v_{i_1}, \ldots, v_{i_k}) = a_{i_1,\ldots,i_k}$.
- Alternating n-tensors on an n-dimensional space.
  - If $\dim V = n$, then $\dim \Lambda^n(V) = 1$, so all n-tensors are multiple of any non-zero one.
  - Hence any $\omega \in \Lambda^n(\mathbb{R}^n)$ is a multiple of det.

- Theorem 4-6: If $v_1, \ldots, v_n$ is a basis for V and $\omega \in \Lambda^n(V)$, then for any n vectors $w_i = \sum_{i=1}^n a_{ij} v_j$, we have $\omega(w_1, \ldots, w_n) = \det(a_{ij}) \cdot \omega(v_1, \ldots, v_n)$.
- For the proof, define $\eta \in \mathcal{T}^n(\mathbb{R}^n)$ by $\eta((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{nn}) = \omega(\sum_j a_{1j} v_j, \ldots, \sum_j a_{nj} v_j)$. Then $\eta \in \Lambda^n(\mathbb{R}^n)$, so $\eta = \lambda \cdot \det$ for some $\lambda \in \mathbb{R}$, and $\lambda = \eta(e_1, \ldots, e_n) = \omega(v_1, \ldots, v_n)$.
- (Alternatively, since V is isomorphic to $\mathbb{R}^n$, it suffices to prove it in the latter case, in which case we know $\omega = c \cdot \det$ for some $c \in \mathbb{R}$, and in this case the claim follows right away if we assume the multiplicativity of the determinant: $\det(A \cdot B) = \det(A) \cdot \det(B)$.)

Exercises:

- The above basic facts about the wedge product
- Spivak 4-1

Problems. **4-1.*** Let $e_1, \ldots, e_n$ be the usual basis of $\mathbf{R}^n$ and let $\varphi_1, \ldots, \varphi_n$ be the dual basis.

(a) Show that $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(e_{i_1}, \ldots, e_{i_k}) = 1$. What would the right side be if the factor $(k+l)!/k!l!$ did not appear in the definition of $\wedge$?

(b) Show that $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_1, \ldots, v_k)$ is the determinant of the $k \times k$ minor of $\begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_k \end{pmatrix}$ obtained by selecting columns $i_1, \ldots, i_k$.

- More generally, show that for any $\varphi_1, \ldots, \varphi_k \in V^*$ and any $v_1, \ldots, v_k \in V$, we have $\omega_1 \wedge \ldots \wedge \omega_k(v_1, \ldots, v_k) = \det\left([\varphi_i(v_j)]_{i,j=1}^k\right)$.

Math 435: Lecture 15

February 12, 2024

Reference: Spivak, pp. 86-89

Topics:

- Orientations
  - By the theorem, any non-zero $\omega \in \Lambda^n(V)$ satisfies $\omega(v_1, \dots, v_n)$ for any basis $v_1, \dots, v_n$ (since by definition we must have $\omega(w_1, \dots, w_n)$ for *some* vectors $w_1, \dots, w_n$).
  - Thus, the set of bases of V is split into two subsets: those with $\omega(v_1, \dots, v_n) > 0$ and those with $\omega(v_1, \dots, v_n) < 0$.
  - Given two bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$ with $w_i \sum_j a_{ij} v_j$, they will be in the same subset if and only if $\det(a_{ij}) > 0$.
  - This condition is independent of ω and always separates the bases of V into two subsets; each of these subsets is called an *orientation* of V.
  - For a basis $v_1, \dots, v_n$, we write $[v_1, \dots, v_n]$ for the orientation to which it belongs, and the other orientation is denoted $-[v_1, \dots, v_n]$.
  - On $\mathbb{R}^n$, we define the *usual orientation* (or *standard orientation*) to be $[e_1, \dots, e_n]$.
- Volume element
  - The characterization of $\det \in \Lambda^n(\mathbb{R}^n)$ by the property $\det(e_1, \dots, e_n) = 1$ is not available in a general vector space V since there is no "standard basis".
  - But now suppose V has an inner product T, and consider orthonormal bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$.
  - If $w_i = \sum_j a_{ij} v_j$, then $A = (a_{ij})$ is an orthogonal matrix: $\delta_{ij} = T(w_i, w_j) = \sum_k a_{ik} a_{jk}$, or in other words $A^\top = A^{-1}$; hence $\det A = \pm 1$.
  - Thus, if $\omega(v_1, \dots, v_n) = \pm 1$, then $\omega(w_1, \dots, w_n) = \pm 1$.
  - If we moreover have an orientation μ, then there is a unique ω with $\omega(v_1, \dots, v_n) = 1$ for an orthonormal oriented basis $[v_1, \dots, v_n] = \mu$.
  - This is called the *volume element* of V determined by T and μ.
  - In $\mathbb{R}^n$, det is the volume element determined by the standard inner product and orientation.
  - The name comes from the fact that $\det(v_1, \dots, v_n)$ is the volume of the parallelepiped spanned by $v_1, \dots, v_n$.
- Tangent vectors
  - For fixed $p \in \mathbb{R}^n$, the set of all pairs (p, v) with $v \in \mathbb{R}^n$ is denoted $\mathbb{R}^n_p$ and is called the *tangent space to* $\mathbb{R}^n$ *at* p.
  - Of course, $\mathbb{R}^n_p$ is in bijection with $\mathbb{R}^n$ itself, and therefore is a vector space (and has a standard basis, standard inner product, standard orientation, etc.).
  - We write v_p for (p, v).
  - The *endpoint* of v_p is the point $p + v$.
- Vector fields
  - A *vector field* on $\mathbb{R}^n$ is a function F on $\mathbb{R}^n$ such that $F(p) \in \mathbb{R}^n_p$ for each p.
  - The *component functions* $F^1, \dots, F^n \colon \mathbb{R}^n \to \mathbb{R}$ of F are given by $F(p) = F^1(p) \cdot (e_1)_p + \cdots + F^n(p) \cdot (e_n)_p$.
  - We say that F is a $\mathcal{C}^k$ vector field if each F^i is a $\mathcal{C}^k$ function.

- Given vector fields F, G and a smooth function f, we can define vector fields $F+G$ and $f \cdot F$, and a function $\langle F, G\rangle$ pointwise: $(F+G)(p) = F(p) + G(p)$ and so on.

- Divergence and curl
  - The *divergence* $\text{div}F$ of a vector field is $\sum_{i=1}^n D_i F^i$.
  - If we consider the "vector of differential operators" $\nabla = \sum_{i=1}^n D_i \cdot e_i = (D_1, \ldots, D_n)$, we can write $\text{div}F = \langle \nabla, F\rangle$.
  - Similarly, in $\mathbb{R}^3$, we can write $(\nabla \times F)$; this is called the *curl* of F.
  - The names "divergence" and "curl" comes from physics you may have seen; we will discuss it later.
- Differential forms
  - A *differential form of degree* k (or just k*-form*) on $\mathbb{R}^n$ is a function ω with $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$ for each $p \in \mathbb{R}^n$.
  - If $\varphi_1(p), \ldots, \varphi_n(p)$ is dual basis to $(e_1)_p, \ldots, (e_n)_p$, we can write

$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\ldots,i_k}(p) \cdot [\varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p)]$$

  - We say that ω is a $\mathcal{C}^l$ k-form if each $\omega_{i_1,\ldots,i_k}$ is $\mathcal{C}^l$. As usual, we only really care about the case $l = \infty$.
  - We can define the sum $\omega + \eta$, multiple $f \cdot \omega$, and $\omega \wedge \eta$ of forms pointwise: $(\omega+\eta)(p) = \omega(p) + \eta(p)$ and so on.
  - We also consider a function as a 0-form and write $f \wedge \omega$ for $f \cdot \omega$.

Exercises:

- Prove (2)-(4) of Theorem 4-8
- Spivak 4-14

4-14. Let c be a differentiable curve in $\mathbf{R}^n$, that is, a differentiable function c: $[0,1] \to \mathbf{R}^n$. Define the tangent vector v of c at t as $c_*((e_1)_t) = ((c^1)'(t), \ldots, (c^n)'(t))_{c(t)}$. If f: $\mathbf{R}^n \to \mathbf{R}^m$, show that the tangent vector to $f \circ c$ at t is $f_*(v)$.

Math 435: Lecture 16

February 14, 2024

Reference: Spivak, pp. 89-90

Topics:

- Differential of a function
  - If $f\colon \mathbb{R}^n \to \mathbb{R}$, we have $\mathrm{D}f(p) \in \Lambda^1(\mathbb{R}^n)$; using the correspondence $\mathbb{R}^n_p \cong \mathbb{R}^n$, we thus obtain a 1-form $\mathrm{d}f$:
  $$\mathrm{d}f(p)(v_p) = \mathrm{D}f(p)(v).$$
  - We write $x^i\colon \mathbb{R}^n \to \mathbb{R}$ for the function returning the i-th coordinate of a point.
  - Warning: there is potential for confusion: sometimes, we write $(x^1, \dots, x^n)$ for the coordinates of a *given* point, so each x^i is a number. Now, we are writing x^i for the coordinate *function*, so $x^i(a^1, \dots, a^n) = a^i$.
  - We now consider the 1-form $\mathrm{d}x^i$. We have
  $$\mathrm{d}x^i(p)(v_p) = \mathrm{D}x^i(p)(v) = v^i.$$
  - Hence $\mathrm{d}x^1(p), \dots, \mathrm{d}x^n(p)$ is the dual basis to $(e_1)_p, \dots, (e_n)_p$, and we can write any k-form ω as
  $$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1,\dots,i_k}(p)\,\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}.$$
  - Theorem 4-7: if $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable, then
  $$\mathrm{d}f = \frac{\partial f}{\partial x^1} \cdot \mathrm{d}x^1 + \dots + \frac{\partial f}{\partial x^n} \cdot \mathrm{d}x^n.$$
  The proof: $\mathrm{d}f(p)(v_p) = \mathrm{D}f(p)(v) = \sum_{i=1}^n v^i \cdot \frac{\partial f}{\partial x^i}(p) = \sum_{i=1}^n \mathrm{d}x^i(p)(v_p) \cdot \frac{\partial f}{\partial x^i}(p)$.
- Pullbacks of differential forms
  - Given a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, we have $\mathrm{D}f(p)\colon \mathbb{R}^n \to \mathbb{R}^m$; thus identifying $\mathbb{R}^n, \mathbb{R}^m$ with $\mathbb{R}^n_p, \mathbb{R}^m_p$, we obtain a linear map $f_*\colon \mathbb{R}^n_p \to \mathbb{R}^m_p$:
  $$f_*(v_p) = \big(\mathrm{D}f(p)(v)\big)_{f(p)}$$
  (In an exercise, you will prove an important alternative characterization of f_*: each $v \in \mathbb{R}^n_p$ is the tangent vector $\gamma'(0)$ to a curve γ through p; and $f_*v \in \mathbb{R}^m_{f(p)}$ is just the tangent vector $(f \circ \gamma)'(p)$ of $f \circ \gamma$ through $f(p)$.)
  - We define the *pullback* of a k-form ω on $\mathbb{R}^m$ to be k-form $f^*\omega$ on $\mathbb{R}^n$ given by $(f^*\omega)(v_1, \dots, v_k) = \omega\big(f(p)\big)(f_*v_1, \dots, f_*v_k)$.
  - Theorem 4-8:
  (1) $f^*\,\mathrm{d}x^i = \mathrm{d}f^i$ (and more generally: $f^*\,\mathrm{d}g = \mathrm{d}(g \circ f)$)
  (2) $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$
  (3) $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$
  (4) $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$
  - Part (1) follows from the chain rule
  - **Exercise:** Prove (2)-(4)
  - Example: suppose we have open sets $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^3$, a differential form α defined on V, and a smooth function $f\colon U \to V$.

– It is helpful to use different notation for the coordinates on U and V; let us write $x, y\colon U \to \mathbb{R}$ for the coordinate functions on U and $X, Y, Z\colon V \to \mathbb{R}$ for the coordinates on V.
Now suppose

$$\alpha = P(X,Y,Z)\,\mathrm{d}X \wedge \mathrm{d}Y + Q(X,Y,Z)\,\mathrm{d}Y \wedge \mathrm{d}Z$$

Then

$$\begin{aligned} f^*\alpha &= (P \circ f)[f^*(\mathrm{d}X^1) \wedge f^*(\mathrm{d}X^2)] + (Q \circ f)[f^*(\mathrm{d}X^2) \wedge f^*(\mathrm{d}X^3)] \\ &= (P \circ f)[\mathrm{d}f^1 \wedge \mathrm{d}f^2] + (Q \circ f)[\mathrm{d}f^2 \wedge \mathrm{d}f^3] \end{aligned} \tag{1}$$

– What this means in practice is that we "express X, Y, Z in terms of x, y and substitute".
Example: suppose

$$\begin{aligned} f(x,y) &= (x \sin y, xy, ye^x) \\ P(X,Y,Z) &= X + Y \\ Q(X,Y,Z) &= X + Z \end{aligned}$$

so that

$$\alpha = (X+Y)\,\mathrm{d}X \wedge \mathrm{d}Y + (X+Z)\,\mathrm{d}Y \wedge \mathrm{d}Z.$$

Now we perform the substitution $X = f^1(x,y) = x \sin y$, $Y = f^2(x,y) = xy$, $Z = f^3(x,y) = ye^x$, and obtain:

$$f^*\alpha = (x \sin y + xy)\,\mathrm{d}(x \sin y) \wedge \mathrm{d}(xy) + (x \sin y + ye^x)\,\mathrm{d}(xy) \wedge \mathrm{d}(ye^x).$$

Notice how this corresponds precisely to the expression (1).
To finish working this out, we should compute each of the differentials. For example:

$$\begin{aligned} \mathrm{d}(x \sin y) \wedge \mathrm{d}(xy) &= (\sin y\,\mathrm{d}x + x \cos y\,\mathrm{d}y) \wedge (x\,\mathrm{d}y + y\,\mathrm{d}x) \\ &= x \sin y\,\mathrm{d}x \wedge \mathrm{d}y + xy \cos y\,\mathrm{d}y \wedge \mathrm{d}x \\ &= (x \sin y - xy \cos y)\,\mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

Math 435: Lecture 17

February 16, 2024

Reference: Spivak, pp. 90-91 and 100

Topics:

- Pulling back n-forms on $\mathbb{R}^n$
  - Theorem 4-9: if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, then
  $$f^*(h\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = (h \circ f)(\det f')(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n).$$
  - To prove it, fix $p \in \mathbb{R}^n$ and let $A = (a_{ij}) = f'(p)$. We then have (omitting "p" from the notation after the first equality sign):
  $$\begin{aligned} f^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n)_p(e_1(p), \ldots, e_n(p)) &= \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n(f_*e_1, \ldots, f_*e_n) \\ &= \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\Big(\sum_{i=1}^n a_{i1}e_i, \ldots, \sum_{i=1}^n a_{in}e_i\Big) \\ &= \det(a_{ij})\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n(e_1, \ldots, e_n) \end{aligned}$$
- Integrating n-forms on $\mathbb{R}^n$ (Spivak p. 100)
  - Let α be a (smooth) n-form defined on an open set $U \subset \mathbb{R}^n$.
    We have $\alpha = h\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ for a unique smooth function h.
    Suppose (for simplicity) that h is bounded. We define:
  $$\int_U \alpha = \int_U h\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n := \int_U h = \int_U h(x^1, \ldots, x^n)\,\mathrm{d}x^1 \ldots \mathrm{d}x^n$$
  - This seems completely trivial: an n form on $\mathbb{R}^n$ is essentially a function, and we define the integral of the n-form.
  - However, we have the following fundamental fact:
    If $V \subset \mathbb{R}^n$ is another open set and $g\colon V \to U$ is an orientation-preserving diffeomorphism (meaning that $\det g'(p) > 0$ for all $p \in V$), then $\int_V g^*\alpha = \int_U \alpha$. Indeed, we have (using $\det g' = |\det g'|$)
  $$\begin{aligned} \int_V g^*\alpha &= \int_V g^*(h\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) \\ &= \int_V (h \circ g)\det g'\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= \int_V (h \circ g)\det g' \\ &= \int_U h \\ &= \int_U \alpha. \end{aligned}$$
  - Thus the formalism of differential forms has the change of variables formula "built in".
  - The differential form α on U is thus a "geometric" object on U which is independent of the particular way U is parametrized (as long as the orientation is preserved).
- Let us write $\Omega^k(U)$ for the set of differential k-forms defined on an open set $U \subset \mathbb{R}^n$.

- Exterior derivative
  - We now generalize the operation $\mathrm{d}\colon \Omega^0(U) \to \Omega^1(U)$ to an operation $\mathrm{d}\colon \Omega^k(U) \to \Omega^{k+1}(U)$ for all k.
  - If
  $$\omega = \sum_{i_1<\dots,i_k} \omega_{i_1,\dots,i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$
  then we define
  $$\begin{aligned}\mathrm{d}\omega &:= \sum_{i_1<\dots,i_k} \mathrm{d}\omega_{i_1,\dots,i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\ &= \sum_{i_1<\dots,i_k} \sum_{j=1}^{n} \frac{\partial}{\partial x^j}\omega_{i_1,\dots,i_k}\,\mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},\end{aligned}$$
  - Theorem 4-10: (1) **Exercise:** $\mathrm{d}(\omega+\eta) = \mathrm{d}\omega + \mathrm{d}\eta$
  (2) **Exercise:** If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, then $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$.
  (This is called the "graded Leibniz rule".)
  (3) $\mathrm{d}(\mathrm{d}\omega)$.
  Briefly, $\mathrm{d}^2 = 0$. (4) If $f\colon U \to V$ is smooth and $\omega \in \Omega^k(U)$, then $f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$.

Exercises:

- $\mathrm{d}(\omega+\eta) = \mathrm{d}\omega + \mathrm{d}\eta$

Math 435: Lecture 18

February 21, 2024

Reference: Spivak, pp. 91-93

Topics:

- Basic properties of the exterior derivative
  - Theorem 4-10: (1) **Exercise:** $\mathrm{d}(\omega + \eta) = \mathrm{d}\omega + \mathrm{d}\eta$
    (2) **Exercise:** If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, then $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$.
    (This is called the "graded Leibniz rule".)
    (3) $\mathrm{d}(\mathrm{d}\omega)$. Briefly, $\mathrm{d}^2 = 0$.
    (4) If $f \colon U \to V$ is smooth and $\omega \in \Omega^k(U)$, then $f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$.
  - For (3), we have (writing I as a shorthand for $i_1 < \ldots < i_k$, ω_I for $\omega_{i_1,\ldots,i_k}$, and $\mathrm{d}x^I$ for $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$:
    $$\mathrm{d}\,\mathrm{d}\omega = \mathrm{d}\sum_I \sum_j \frac{\partial}{\partial x^j}\omega_I \,\mathrm{d}x^j \wedge \mathrm{d}x^I = \sum_I \sum_{j,k} \frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j}\omega_I \,\mathrm{d}x^k \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I.$$
    But now switching j and k reverses the sign but gives the same result, so the sum must be zero.
  - (4) is proven by induction. We have seen that it's true for 0-forms (by the chain rule). For the induction step, we note that every $(k+1)$-form is a sum of forms of the form $\omega \wedge \mathrm{d}x^i$, and we have
    $$\begin{aligned} f^*\,\mathrm{d}(\omega \wedge \mathrm{d}x^i) &= f^*(\mathrm{d}\omega \wedge \mathrm{d}x^i) \\ &= f^*\,\mathrm{d}\omega \wedge f^*\,\mathrm{d}x^i \\ &= \mathrm{d}(f^*\omega) \wedge \mathrm{d}f^i \\ &= \mathrm{d}(f^*\omega \wedge \mathrm{d}f^i) \\ &= \mathrm{d}(f^*\omega \wedge f^*\,\mathrm{d}x^i) \\ &= \mathrm{d}f^*(\omega \wedge \mathrm{d}x^i) \end{aligned}$$
- Closed and exact forms
  - A form $\omega \in \Omega^k(U)$ is *closed* if $\mathrm{d}\omega = 0$ and *exact* if $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$.
  - The above theorem shows that every exact form is closed.
  - We ask: is every exact form closed?
  - The answer is: *yes* for forms on $\mathbb{R}^n$, but *no* for forms on general open subsets U.
  - Example: let us write $x = r\cos\theta$ and $y = r\sin\theta$.
    Then $\mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta$ and $\mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta$.
    Solving for $\mathrm{d}\theta$, we obtain
    $$\mathrm{d}\theta = \frac{-r\sin\theta\,\mathrm{d}x}{r^2} + \frac{r\cos\theta\,\mathrm{d}y}{r^2} = \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$
  - Note that $\mathrm{d}\theta$ is defined on $U = \mathbb{R}^2 - \{(0,0)\}$.
  - Despite the notation, $\mathrm{d}\theta$ is *not* exact. The point is that θ is not actually a well-defined function on all of $\mathbb{R}^2 - \{0\}$; it is defined on $U_1 = \mathbb{R}^2 - \{(x,y) \mid x \leq 0\}$ (and takes values in $(0, 2\pi)$) and it is defined on $U_2 = \mathbb{R}^2 - \{(x,y) \mid x \geq 0\}$ (and takes values in $(-\pi,\pi)$). Hence on U_1 and U_2, $\mathrm{d}\theta$ *is* an exact form; but on U it is *not*.

- However, it is *closed*, since it is exact on U_1 and U_2, hence $\mathrm{d}\,\mathrm{d}\theta(p) = 0$ for each point of $U = U_1 \cup U_2$.
- To see it is not exact, we need the general theory of integration of forms, which we don't have yet. Let us sketch the argument, as a "preview" of how this theory works. Given a curve $\gamma\colon [a, b] \to \mathbb{R}^2$, we can integrate any 1-form α over γ: $\int_\gamma \alpha$. As one might guess, for an exact form $\mathrm{d}f$, we have $\int_\gamma \mathrm{d}f = f(b) - f(a)$. In particular, if $\gamma(a) = \gamma(b)$, then $\int_\gamma \mathrm{d}f = 0$.

  Now let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Let us try to compute $\int_\gamma \mathrm{d}\theta$ (even though we do not yet know what this means!); we proceed "formally". We have $x = \cos t$ and $y = \sin t$, and hence $\mathrm{d}x = -\sin t\,\mathrm{d}t$ and $y = \cos t\,\mathrm{d}t$. Thus

  $$\int_\gamma \frac{-y}{x^2 + y^2}\,\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y = \int_0^{2\pi} \sin^2 t\,\mathrm{d}t + \cos^2 \mathrm{d}t = 2\pi$$

  Since this is not zero, γ cannot be exact.
- The amazing thing about the theory of integrating differential forms is that it makes the above "symbolic" computation rigorous.

Exercises:

- Find a differential form $\alpha \in \Omega^{n-1}(\mathbb{R}^n)$ such that $\mathrm{d}\alpha = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. (You might want to start with the case $n = 1$ or $n = 2$.)

Math 435: Lecture 19

February 23, 2024

Reference: Spivak, pp. 93-95 and 109-112

Topics:

- Poincaré lemma
  - We return to the other answer to the question: every closed $\omega \in \Omega^k(\mathbb{R}^n)$ *is exact.*
  - More generally, an open set $U \subset \mathbb{R}^n$ is *star shaped* if for each $p \in U$, the line segment $\{t \cdot p \mid 0 \le t \le 1\}$ is contained in U.
  - Theorem 4-11 (Poincaré Lemma): If $U \subset \mathbb{R}^n$ is star-shaped, then every closed $\omega \in \Omega^k(U)$ is exact (for $k \ge 1$).
  - Here is the proof when $k = 1$.
  - Let $\omega = \sum_{i=1}^n \omega_i \, \mathrm{d}x^i$. We have $\mathrm{d}\omega = \sum_{i,j=1}^n \mathrm{D}_j\omega_i \, \mathrm{d}x^j \wedge \mathrm{d}x^i = \sum_{i<j} (\mathrm{D}_i\omega_j - \mathrm{D}_j\omega_i) \, \mathrm{d}x^i \wedge \mathrm{d}x^j$.

    Hence ω is closed if and only if $\mathrm{D}_i\omega_j = \mathrm{D}_j\omega_i$ for all $i < j$.
  - Next suppose ω *were* exact, so $\omega = \mathrm{d}f$ and $\omega_i = \mathrm{D}_i f$ for all i.
  - Then (since U is star-shaped!) $f(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(f(tx)) \, \mathrm{d}t = \int_0^1 \sum_{i=1}^n \mathrm{D}_i f(tx) x^i \, \mathrm{d}t = \int_0^1 \sum_{i=1}^n \omega_i(tx) x^i \, \mathrm{d}t$.
  - Hence, given an arbitrary closed ω, we define f by the above formula.
  - Then (using $\mathrm{D}_i\omega_j = \mathrm{D}_j\omega_i$), we have

$$\begin{aligned}
\mathrm{D}_j f(x) &= \int_0^1 \sum_{i=1}^n [\mathrm{D}_j\omega_i(tx) t x^i + \omega_i(tx)\delta_{ij}] \, \mathrm{d}t \\
&= \int_0^1 \Big[\sum_{i=1}^n \big[\mathrm{D}_i\omega_j(tx) t x^i\big] + \omega_j(tx) \Big] \, \mathrm{d}t \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \big[\omega_j(tx) t\big] \, \mathrm{d}t \\
&= \omega_j(x).
\end{aligned}$$

- Manifolds
  - A subset $M \subset \mathbb{R}^n$ is a *(smooth) k-dimensional submanifold of* $\mathbb{R}^n$ (or just *k-manifold*) if for each $x \in M$, there is an open subset $U \subset \mathbb{R}^n$ containing x, an open subset $V \subset \mathbb{R}^n$, and a diffeomorphism $h \colon U \to V$ such that

$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V \mid y^{k+1} = \cdots = y^n = 0\}.$$

    (One can also speak of $\mathcal{C}^l$-manifolds for any $l \ge 0$ by demanding that the charts h be $\mathcal{C}^l$-diffeomorphisms rather than smooth.)
  - Some terminology: The map h is called a *flattener*.

    By restriction, we obtain an induced map $\varphi \colon \langle h^1, \ldots, h^k \rangle \colon U \cap M \to W \subset \mathbb{R}^k$ where $W = \{y \in \mathbb{R}^k \mid (y, 0) \in V\}$, called a *(k-dimensional) coordinate chart* (or just a *chart*) for M.

    The inverse, $\varphi^{-1} \colon W \to U \cap M$ is called a *parametrization* of $U \cap M$; it is smooth, since it is the composition of the smooth map h^{-1} with the embedding $W \to V$ ($y \mapsto (y, 0)$).

    A *(k-dimensional) atlas* for M is a set of coordinate charts $\{h_\alpha \colon U_\alpha \cap M \to W\}_{\alpha \in I}$

such that the sets U_α cover M.
Thus, a subset $M \subset \mathbb{R}^n$ is a k-manifold if and only if it admits a k-dimensional atlas.
  - Example: the n-sphere $\mathrm{S}^n = \{x \mid \mathbb{R}^{n+1} \mid |x| = 1\}$.
    This can be checked directly or with the following theorem.
- Manifolds as zero-sets
  - A point $x \in U$ of an open subset $U \subset \mathbb{R}^n$ is a *regular point* of a smooth map $g\colon U \to \mathbb{R}^p$ is $\mathrm{D}g(x)\colon \mathbb{R}^n \to \mathbb{R}^p$ is surjective (or equivalently, if $g'(x)$ has rank p); otherwise, it is called a *critical point*.
    A point $a \in \mathbb{R}^p$ in called a *regular value* of g if x is a regular point of g for all $x \in g^{-1}(a)$; otherwise it is called a *critical value*.
  - Theorem 5-1: if $U \subset \mathbb{R}^n$ is open and $g\colon U \to \mathbb{R}^p$ is smooth and 0 is a regular value of g, then $g^{-1}(0)$ is an $(n-p)$-dimensional manifold in $\mathbb{R}^n$.
  - The proof uses the inverse function theorem (and is, together with the next theorem, the most important application of the latter).
    Given $x \in M := g^{-1}(0)$, we must find a flattener for M on a neighbourhood of x.
    Since $\mathrm{D}g(x)\colon \mathbb{R}^n \to \mathbb{R}^p$ is surjective, we have by the rank-nullity theorem that $\ker\big(\mathrm{D}g(x)\big)$ is $n-p$-dimensional.
    By applying an invertible linear transformation to M, we can assume $x = 0$ and $\ker\big(\mathrm{D}g(x)\big) = \mathbb{R}^{n-p} \times \{0\}$ (it's easy to see that the existence of a flattener is unaffected by applying an invertible linear transformation).
    Since $\mathrm{D}g(x)$ is surjective and $\mathrm{D}g(e_i) = 0$ for $i \le n-p$, it follows that $\mathrm{D}g(x)(e_{n-p+1}), \dots, \mathrm{D}g(x)(e_n)$ span $\mathbb{R}^p$. Now let $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-p}$ be the projection onto the first $n-p$ coordinates, and define a map $\tilde{g}\colon A \to \mathbb{R}^{n-p} \times \mathbb{R}^p \cong \mathbb{R}^n$ by $\tilde{g}(y) = \big(\pi(y), g(y)\big)$.
    Since $\mathrm{D}_i g^j(x) = 0$ for $i \le n-p$, we see that $g'(x)$ is a block-diagonal matrix, with one block $(n-p)\times(n-p)$ identity matrix, and the second block having rank p, and thus invertible. It follows that $\det g'(x) \neq 0$.
    Hence, by the inverse function theorem, there is a neighbourhood $U \subset \mathbb{R}^n$ of x and $V \subset \mathbb{R}^n$ of $\tilde{g}(x) = (\pi(x), 0)$ such that $\tilde{g}|_U\colon U \to V$ is a diffeomorphism.
    Since $M = g^{-1}(0)$, it follows that $\tilde{g}(U \cap M) = V \cap (\mathbb{R}^{n-p} \times \{0\})$, and we are done.

Exercises:

- For any smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, find a differential form $\alpha \in \Omega^{n-1}(\mathbb{R}^n)$ such that $\mathrm{d}\alpha = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. (Again, you might want to start with the cases $n = 1, 2$.)

Math 435 lecture plan

Spring 2024

Note: this lecture plan is tentative and will be adjusted to adapt to the pace of the course.

Week 7 (Monday, Feb 19)
Spivak, chapter 4

- *(President's Day)*
- Closed and exact forms
- Poincaré Lemma

Week 8 (Monday, Feb 26)
Spivak, chapter 5

- **Midterm in class on Monday, Feburary 26**
- Manifolds
- Differential forms on manifolds
- Orientations

Week 9 (Monday, Mar 4)
Spivak, chapter 5

- Stokes' theorems on manifolds
- The volume element
- The classical theorems (Divergence, Green's, Stokes')

Week 10 (Monday, Mar 11)
Spring break

Week 11 (Monday, Mar 18)
Tapp, Chapter 1-2

- Curves in $\mathbb{R}^n$
- Arc length
- Curvature
- Torsion
- Curves with prescribed curvature and torsion
- Hopf's Umlaufsatz
- Isoperimetric inequality

Week 12 (Monday, Mar 25)
Tapp, Chapters 3.7-3.9 and 4.1

- The first fundamental form
- Archimedes hat box theorem
- The Gauss map

Week 13 (Monday, Apr 1)
Tapp, Chapters 4.3-4.5

- The second fundamental form
- Normal curvature and Gaussian curvature

Week 14 (Monday, Apr 8)
Tapp, Chapters 5.1-5.3

- Geodesics
- Exponential map
- Theorema Egregium

Week 15 (Monday, Apr 15)
Tapp, Chapter 6

- The Gauss-Bonnet theorem
- Compact surfaces

Week 16 (Monday, Apr 22)

- (Time permitting) Riemann surfaces

Week 17 (Monday, Apr 29)

- (Time permitting) Hyperbolic geometry