MATH 440, TOPOLOGY

Instructor: Joseph Helfer

Fall 2022

Course outline

This is an outline of what we have covered in the semester so far, with main definitions and statements. No proofs!

Version as of: *December 1, 2022*

Contents

MATH 440 NOTES

1. Set theory

Axiom 1 ("extensionality"). Two sets A and B are equal if and only if: for any thing x, x is contained in A if and only if x is contained in B. In symbols:

$$A = B \iff \forall x\ (x \in A \iff x \in B).$$

Axiom 2 ("unrestricted comprehension"). For any property P, there exists a (by extensionality unique) set A – called the *extension of* P – such that for any thing x, x is an element of A if and only if x has the property P.

In symbols: if $\varphi(x)$ is a formula stating that x has some property, then

$$\exists A\ \forall x\ (x \in A \iff \varphi(x)).$$

The usual notation for the set of all x satisfying $\varphi(x)$ is

$$\{x \mid \varphi(x)\}.$$

Thus:

$$\forall a\ (a \in \{x \mid \varphi(x)\} \iff \varphi(a)).$$

1.1. Some basic notions.

Definition 1.1.1.

- $A \subset B$ or $B \supset A$ means $\forall x\ (x \in A \Rightarrow x \in B)$ ("A is a subset of B")
- $A \subsetneq B$ or $B \supsetneq A$ means ($A \subset B$ and $A \neq B$) ("A is a proper subset of B")

(**Theorem**: ($A \subset B$ and $B \subset A$) $\iff$ $A = B$)

- We write $\{a\}$ for the set just containing $\{a\}$ (i.e., $\{a\} = \{x \mid x = a\}$); a set of this form is called a singleton. Similarly, $\{a, b\} = \{x \mid x = a \text{ or } x = b\}$, and $\{a, b, c\} = \{x \mid x = a \text{ or } x = b \text{ or } x = c\}$, and so on.
- We denote by $\varnothing$ or $\{\}$ the empty set, which is characterized by the property that $a \notin \varnothing$ for all things a. We can define it using the comprehension axiom as $\{x \mid x \neq x\}$.

Given sets A and B,

- $A \cup B$ denotes the set $\{x \mid x \in A \text{ or } x \in B\}$, the union of A and B
- $A \cap B$ denotes the set $\{x \mid x \in A \text{ and } x \in B\}$, the intersection of A and B
- $A - B$ or $A \backslash B$ denotes $\{x \mid x \in A \text{ and } x \notin B\}$, the difference of A and B. If $B \subset A$, this is also called the complement of B in A.

A and B are called disjoint if $A \cap B = \varnothing$, i.e., if there are not things which belong to both A and B.

Some set algebra:

Theorem 1.1.2. For any sets A, B, C:
"Associativity":

- $(A \cup B) \cup C = A \cup (B \cup C)$

- $(A \cap B) \cap C = A \cap (B \cap C)$

"Commutativity":

- $A \cup B = B \cup A, \quad A \cap B = B \cap A$

"Distributivity":

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

"De Morgan's laws":

- $A - (B \cup C) = (A - B) \cap (A - C)$
- $A - (B \cap C) = (A - B) \cup (A - C)$

Properties of the empty set:

- $A \cup \varnothing = A, \quad A \cap \varnothing = \varnothing, \quad A - \varnothing = A$

Properties of the "whole" set: if $B \subset A$, then

- $A \cup B = A, \quad A \cap B = B, \quad A - (A - B) = B$

Definition 1.1.3. Given a set A, its power set $\mathcal{P}(A)$ is the set of all subsets of A: $\mathcal{P}(A) = \{B \mid B \subset A\}$.

Definition 1.1.4. Given a *set of sets* $\mathcal{A}$, (i.e., a set, *each of whose elements is a set*), its union, denoted $\bigcup \mathcal{A}$ or $\bigcup_{A \in \mathcal{A}} A$, is the set

$$\bigcup \mathcal{A} = \{x \mid \exists A \ (A \in \mathcal{A} \text{ and } x \in A)\}.$$

Note that it is typical to abbreviate "$\exists x \ (x \in X \text{ and } \varphi(x))$" as "$\exists x \in X \ (\varphi(x))$". So the above would be

$$\bigcup \mathcal{A} = \{x \mid \exists A \in \mathcal{A} \ (x \in A)\}.$$

If $\mathcal{A}$ is not empty, then the intersection of $\mathcal{A}$, denoted $\bigcap \mathcal{A}$ or $\bigcap_{A \in \mathcal{A}} A$ is

$$\bigcap \mathcal{A} = \{x \mid \forall A \ (A \in \mathcal{A} \implies x \in A)\} = \{x \mid \forall A \in \mathcal{A} \ (x \in A)\}.$$

Here, we are using the standard abbreviation "$\forall x \in X \ (\varphi(x))$" of "$\forall x \ (x \in X \implies \varphi(x))$".

1.2. Cartesian products.

Definition 1.2.1. Give any two things a and b, the ordered pair (a, b) of a and b is the set

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Theorem 1.2.2. Given any things a, b, a', b', we have $(a, b) = (a', b')$ iff $a = a'$ and $b = b'$.

Definition 1.2.3. Given sets A and B, their cartesian product (or just product) is the set

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

The above notation is read "all things of the form (a, b) such that $a \in A$ and $b \in B$", and is an abbreviation for

$$\{x \mid \text{There exist } a \text{ and } b \text{ such that } x = (a, b) \text{ and } a \in A \text{ and } b \in B\}.$$

Hence, by definition, we have that $x \in A \times B$ if and only if $x = (a, b)$ for some $a \in A$ and $b \in B$.

1.3. **Functions and relations.**

Definition 1.3.1.

- Given sets A and B, a binary relation (or just relation) between A and B is a subset $R \subset A \times B$ of the cartesian product $A \times B$.
  Given a relation $R \subset A \times B$ and elements $a \in A$ and $b \in B$, we may sometimes write aRb for $(a, b) \in R$.
- A relation $f \subset A \times B$ is a function (or mapping or map) if for each $a \in A$ there is a unique $b \in B$ with $(a, b) \in f$.
  In symbols: $\forall a \in A\ \exists! b \in B\ [(a, b) \in f]$. Here "$\exists!$" is read "there exists a unique".
  In this case, we say that f is a function from A to B or with domain A and codomain B, and write $f\colon A \to B$.
  (Note that what we call "codomain" Munkres calls "range". We use the word "range" for something else (see below).
- Given a function $f \subset A \times B$ and an element $a \in A$, we write $f(a)$ for the unique element $b \in B$ satisfying $(a, b) \in f$, and call it the image of a under f or the value of f at a.
- The range of f is the subset $\{f(a) \mid a \in A\} \subset B$. Note that we are using the same notation as in Definition 1.2.3 on page 4. I.e., it is short for $\{x \mid \exists a \in A(x = f(a))\}$. In the future, we will use this kind of notation without explaining it.

To define a function $f\colon A \to B$ it suffices to specify, for each $a \in A$, what the unique element $b \in B$ is such that $f(a) = b$. Thus, when we say "let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$", we are saying that f is the function

$$f = \{(x, x^2) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}.$$

- Given a function $A \to B$ and a subset $A_0 \subset A$, the restriction of f to A_0, denoted $f|_{A_0}$ is the function $f|_{A_0}\colon A_0 \to B$ defined by the rule $(f|_{A_0})(x) = f(x)$ for $x \in A_0$; i.e., it is the function
$$f|_{A_0} = \{(a, f(a)) \mid a \in A_0\} \subset A \times B.$$
- Given functions $f\colon A \to B$ and $g\colon B \to C$, the composite of f and g, denoted $g \circ f$ is the function $g \circ f\colon A \to C$ defined by the rule $(g \circ f)(a) = g(f(a))$ for $a \in A$.
- For any set A, the identity function on A, denoted id_A is the function id_A
- A function $f\colon A \to B$ is injective or one-to-one if $f(a) = f(a')$ implies $a = a'$ for any $a, a' \in A$. The function f is surjective or onto if for each $b \in B$ there is an $a \in A$ such that $f(a) = b$, and f is bijective if it is both injective and surjective.
  If f is injective, surjective, or bijective, we also say that it is an injection, surjection, or bijection, respectively.
  We often indicate that f is a bijection by writing $f\colon A \xrightarrow{\sim} B$, and similarly, we write $A \cong B$ to indicate that there exists a bijection $A \xrightarrow{\sim} B$.
- If $f\colon A \to B$ is a bijection, the inverse of f, denoted f^{-1}, is the function $f^{-1}\colon B \to A$ such that $f^{-1}(b)$ is the unique element $a \in A$ such that $f(a) = b$.

(**Theorem**: $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.)
(The proof of this theorem uses an important:
Lemma: given functions $F, G\colon X \to Y$, we have $F = G$ if and only if $F(x) = G(x)$ for all $x \in X$.)
(**Theorem**: conversely, if there exists $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, then f is a bijection.)

- Given a function $f\colon A \to B$ and a subset $A_0 \subset A$, the image of A_0 under f, denoted $f(A_0)$, is the subset $\{f(a) \mid a \in A_0\} \subset B$.

  Given $B_0 \subset B$, the pre-image of B_0 under f is the subset $\{a \in A \mid f(a) \in B_0\} \subset A$.

  (Here, we have again introduced a variant of the set builder notation: given a set X and a property $\varphi(x)$ which can be satisfied by elements of X, rather than writing $\{x \mid x \in X \text{ and } \varphi(x)\}$, we will often just write $\{x \in X \mid \varphi(x)\}$.)

WARNING 1.3.2. We are now using the notation f^{-1} for two different things; don't get them confused! Similarly, note the difference between $f(a)$ for $a \in A$ and $f(A_0)$ for $A_0 \subset A$.

1.4. **Equivalence relations and quotients.**

Definition 1.4.1. Given a set A, a binary relation (or just relation) on A is just a binary relation between A and itself – i.e., a subset $R \subset A \times A$.

A relation $R \subset A \times A$ on A is an equivalence relation if it satisfies the following for all $x, y, z \in A$:

(i) (Reflexivity) xRx
(ii) (Symmetry) If xRy, then yRx.
(iii) (Transitivity) If xRy and yRz, then xRz.

It is common practice to use the symbol "$\sim$" to denote an equivalence relation.

Given an equivalence relation $\sim \subset A \times A$ on A and an element of x, the $\sim$-equivalence class of x (or just the equivalence class of x) is the set, often denoted by $[x]_\sim$, or just $[x]$, defined by

$$[x] = \{y \in A \mid y \sim x\}.$$

The quotient of A by $\sim$, denoted $A/{\sim}$, is the set of $\sim$-equivalence classes:

$$A/{\sim} = \{[x] \mid x \in A\}.$$

Given an equivalence class $S \subset A/\sim$, any element $x \in S$ is called a representative of S. For any representative s of S, we have $S = [s]$ (why?).

Lemma 1.4.2. Given an equivalence relation on a set X, any two equivalence classes are disjoint or equal.

Definition 1.4.3. Given a set A, a partition of A is a set $\mathcal{D} \subset \mathcal{P}(A)$ of *nonempty* subsets of A such that

(i) Any two distinct elements of $\mathcal{D}$ are disjoint
(ii) $\bigcup \mathcal{D} = A$

In other words: $\forall a \in A\ \exists! S \in \mathcal{D}\ (a \in S)$.

Theorem 1.4.4. "Partitions are the same as equivalence relations." More precisely:

For any equivalence relation $\sim$ on a set A, the set $A/{\sim}$ of equivalence classes forms a partition of A.

Conversely, every partition $\mathcal{D}$ of A gives rise to an equivalence relation $E_{\mathcal{D}}$ defined by $xE_{\mathcal{D}}y$ iff $x, y \in S$ for some $S \in \mathcal{D}$.

Finally, if $\mathrm{ER}(A)$ is the set of equivalence relations on A and $\mathrm{PT}(A)$ is the set of partition on A, the functions

$$\mathrm{ER}(A) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{PT}(A)$$

defined by $F(E) = A/E$ and $G(\mathcal{D}) = E_{\mathcal{D}}$ form a bijection, i.e., $G = F^{-1}$.

1.5. **Order relations.**

Definition 1.5.1. A relation $R \subset A \times A$ is a partial order if it satisfies the following for all $a, b, c \in A$:

(i) ("Reflexivity") aRa
(ii) ("Transitivity") aRb and bRc implies aRc
(iii) ("Anti-symmetry") aRb and bRa implies $a = b$

A set A together with a partial order $R \subset A \times A$ is called a partially-ordered set or poset. It's common to denote an partial order by $\leqslant \subset A \times A$, in which case we write $a < b$ for ($a \leqslant b$ and $a \neq b$). We also say "a is less than b" for $a < b$, and "a is less than or equal to b" (or "a is at most b") for $a \leqslant b$. We also write $b \geqslant a$ for $a \leqslant b$ and $b > a$ for $a < b$.

Definition 1.5.2. A partial order $\leqslant \subset A \times A$ is a total order or linear order if it additionally satisfies: for any $a, b \in A$, either $a \leqslant b$ or $b \leqslant a$.

A set together with a total order is called a totally (or linearly) ordered set.

Remark 1.5.3. Note that one can instead axiomatize the relation "$<$" associated to a partial order, which is called a *strict partial order*: we simply replace (i) "reflexivity" with "anti-reflexivity": $\forall a \in A\ \neg(aRa)$ (the symbol $\neg$ means "not"). (This is what Munkres does.) (And a "strict total order" is then defined to be a strict partial order such that aRb or bRa for all a, b *such that* $a \neq b$.)

Definition 1.5.4. Given a total order $\leqslant$ on A and $a, b \in A$, we define the open, closed, and half-open intervals between a and b, respectively as:

$$\begin{matrix} [a,b] \\ (a,b) \\ (a,b] \\ [a,b) \end{matrix} = \left\{ x \in X \,\middle|\, a \begin{matrix} \leqslant \\ < \\ < \\ \leqslant \end{matrix}\ x\ \begin{matrix} \leqslant \\ < \\ \leqslant \\ < \end{matrix}\ b \right\}$$

and similarly, we write $[a, \infty)$ for $\{x \in X \mid x \geqslant a\}$ and $(-\infty, a)$ for $\{x \in X \mid x < a\}$, and so on.

WARNING 1.5.5. We are now using (a, b) to denote both an ordered pair (which is defined for *any* things a and b) as well as an open interval (which is also defined for a and b elements of a given totally ordered set). It should always be clear from context which one is meant.

1.6. **The natural numbers and "everything is a set".**

Note that the following treatment of the natural and real numbers is somewhat different from Munkres'.

Theorem 1.6.1 ("Dedekind-Peano axioms")**.** There exists a set $\mathbb{N}$, an element $0 \in \mathbb{N}$ (which we call "zero"), and a function $S \colon \mathbb{N} \to \mathbb{N}$ (which we call the "successor function"), satisfying the following properties:

(i) Zero is not a successor; i.e., there is no $n \in \mathbb{N}$ with $Sn = 0$
(ii) The successor function is injective; i.e., given $m, n \in \mathbb{N}$ with $Sm = Sn$, then $m = n$
(iii) ("The principle of mathematical induction"): given a subset $M \subset \mathbb{N}$, if $0 \in M$ and, for each $n \in M$, we have $Sn \in M$, then $M = \mathbb{N}$. In symbols:

$$\forall M \subset \mathbb{N}\Bigg(\Big(0 \in M \wedge \forall n \in \mathbb{N}(n \in M \Rightarrow Sn \in M)\Big) \Rightarrow M = \mathbb{N}\Bigg)$$

Property (iii) is typically employed as follows: we want to prove that every $n \in \mathbb{N}$ has some property P. We let M be the set of all $n \in \mathbb{N}$ satisfying P. We then show that 0 satisfies P, and that if n satisfies P, so does Sn. It then follows from property (iii) that $M = \mathbb{N}$, and hence every n satisfies P. This is called a "proof by induction".

Definition 1.6.2. We call a triple $(\mathbb{N}, 0, S)$ satisfying the conditions in the above theorem a "Dedekind system".

(We can define ordered triples as $(a, b, c) := (a, (b, c))$; using Theorem 1.2.2, it is easy to prove the fundamental property $(a, b, c) = (a', b', c')$ iff ($a = a'$ and $b = b'$ and $c = c'$).)

Given a Dedekind system $(\mathbb{N}, 0, S)$, we usually write 1 for $S0$, and then 2 for $S1 = S(S0)$, and so on.

Theorem 1.6.3 ("Recursion theorem")**.** Let $(\mathbb{N}, 0, S)$ be a Dedekind system and A a set. Given an element $a \in A$ and a function $g\colon A \to A$, there is a unique function $f\colon \mathbb{N} \to A$ satisfying:

(i) $f(0) = a$
(ii) $f(Sn) = g(f(n))$ for all $n \in \mathbb{N}$.

When we apply the recursion theorem to prove the existence of a function f, it is called "defining f by recursion". Here is an example:

Definition 1.6.4. Let $(\mathbb{N}, 0, S)$ be a Dedekind system.

Fix some $n \in \mathbb{N}$. Let us apply the recursion theorem, where we take $A = \mathbb{N}$, $a = n$, and $g = S$. Thus, the theorem says that there exists a unique function $f_n\colon \mathbb{N} \to \mathbb{N}$ satisfying $f_n(0) = a = n$ and $f_n(Sm) = g(f_n(m)) = S(f_n(m))$ for each $m \in \mathbb{N}$.

(Normally, we would just say "we define the function $f_n\colon \mathbb{N} \to \mathbb{N}$ by *recursion* (or *recursively*, or by *induction*) by setting $f_n(0) = 0$ and $f_n(Sm) = S(f_n(m))$ for $m \in \mathbb{N}$".)

We write $n + m$ for $f_n(m)$. Thus, we have by definition that $n + 0 = n$ and $n + Sm = S(n + m)$.

Note that $n + 1 = Sn$ for any $n \in \mathbb{N}$, so we will often write $n + 1$ in place of Sn.

Definition 1.6.5. Again fix a Dedekind system $(\mathbb{N}, 0, S)$.

For each $a \in \mathbb{N}$, we define $m_a\colon \mathbb{N} \to \mathbb{N}$ recursively by setting $m_a(0) = 0$ and $m_a(b+1) = m_a(b) + a$ for $b \in \mathbb{N}$. We write $a \cdot b$ (or just ab) for $m_a(b)$. (Thus, by definition, we have $a \cdot 0 = 0$ and $a \cdot (b + 1) = a \cdot b + a$ – where we employ the usual convention that the operation "$\cdot$" has higher precedence than "+".

Similarly, for each $a \in \mathbb{N} - \{0\}$, we define $e_a\colon \mathbb{N} \to \mathbb{N}$ recursively by setting $e_a(0) = 1$ and $e_a(b + 1) = e_a(b) \cdot a$; and then define $a^b = e_a(b)$. We then have $a^0 = 1$ and $a^{b+1} = a^b \cdot a$.

Some basic properties (all of which are proven by induction):

Theorem 1.6.6. Again fix a Dedekind system $(\mathbb{N}, 0, S)$. For any $a, b, c \in \mathbb{N}$:
"Associativity":

- $(a + b) + c = a + (b + c), \quad (ab)c = a(bc)$

"Commutativity":

- $a + b = b + a, \quad ab = ba$

"Distributivity":

- $a(b + c) = ab + ac$

"Exponential laws":

- $a^{b+c} = a^b \cdot a^c, \quad a^{b \cdot c} = (a^b)^c$

Definition 1.6.7. Again fix a Dedekind system $(\mathbb{N}, 0, S)$. We define a total ordering $\leqslant \subset \mathbb{N} \times \mathbb{N}$ by declaring that $a \leqslant b$ if and only if $a + d = b$ for some $d \in \mathbb{N}$.

(**Theorem**: This is a total ordering on $\mathbb{N}$.)

Theorem 1.6.8. Any two Dedekind systems are "isomorphic":

Given any two Dedekind systems $(\mathbb{N}, 0, S)$ and $(\mathbb{N}', 0', S')$, there is a (in fact, unique) bijection $f: \mathbb{N} \to \mathbb{N}'$ such that $f(0) = 0'$ and $f(Sn) = S'(f(n))$ for all $n \in \mathbb{N}$.

We now turn to the construction of a Dedekind system.

Definition 1.6.9. A set X is *inductive* if $\varnothing \in X$ and whenever $A \in X$ for some set A, we have $A \cup \{A\} \in X$.

We define the set of *von Neumann natural numbers*, denoted ω, to be the smallest inductive set; i.e., it is the set of all things contained in every inductive set:

$$\{x \mid \forall X \text{ (if } X \text{ is an inductive set then } x \in X)\}$$

We define a function $S_\omega: \omega \to \omega$ by setting $S_\omega(n) = n \cup \{n\}$.

The following is a more precise version of Theorem 1.6.1:

Theorem 1.6.10. $(\omega, \varnothing, S_\omega)$ is a Dedekind system.

From now on, for definiteness, we no longer use $(\mathbb{N}, 0, S)$ to denote an arbitrary Dedekind system, but the particular one given by the von Neumann natural numbers; that is, we define $\mathbb{N} = \omega$, $0 = \varnothing$ and $S = S_\omega$. (However, in light of Theorem 1.6.8, it "doesn't matter" which Dedekind system we use, and therefore you should "forget" about the von Neumann natural numbers and just rely on the Dedekind-Peano axioms.)

Remark 1.6.11. Emboldened by this success, we might be inclined to adopt the

Axiom: every thing is a set.

(One can also give a formal definition of "pure set" – i.e., a set all of whose elements are sets, each of whose sets are also sets, "etc." – and then go even further and adopt the axiom "every thing is a pure set"; this is called the *axiom of foundation*.)

We could indeed adopt such an axiom, as it will continue to be the case that we do not need anything but pure sets for doing all of mathematics – and indeed, people do often adopt this axiom in certain contexts. However, for us, there is nothing really to gain from this axiom except a sense of comfort. All the things we use will be pure sets anyway, and it doesn't make a difference if we declare that there aren't any other things.

Definition 1.6.12. Some notation: we denote by $\mathbb{N}_+$ or $\mathbb{Z}_+$ or $\mathbb{N}_{>0}$ or $\mathbb{Z}_{>0}$ the set $\mathbb{N}\backslash\{0\}$. We may also write $\mathbb{Z}_{\geqslant 0}$ for $\mathbb{N}$.

Given $a, b \in \mathbb{N}$ with $a \leqslant b$, we denote by $\{a, \ldots, b\}$ the set $\{n \in \mathbb{N} \mid a \leqslant n \leqslant b\}$. If $a > b$, then $\{a, \ldots, b\} = \varnothing$.

1.7. **The real numbers.**

Definition 1.7.1. A binary operation on a set X is a function $f: X \times X \to X$. Thus, for each $x, y \in X$, this gives us an element $f(x, y) \in X$.

Often, we denote a binary function by a symbol such as "$+: X \times X \to X$", in which case we usually use "infix" notation and write $x + y$ instead of $+(x, y)$. Often, if the symbol happens to be "$\cdot$", we moreover just omit it and write xy instead of $x \cdot y$.

Definition 1.7.2. A field is a triple $(F, +, \cdot)$, where F is a set and $+$ and $\cdot$ are binary operations on F (called "addition" and "multiplication"), satisfying:

(i) Both operations are commutative and associative, i.e., for any $a, b, c \in F$:

$$a + b = b + a, \quad a \cdot b = b \cdot a, \quad (a + b) + c = a + (b + c), \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

(ii) There is a unique element $0 \in K$ satisfying $0 + a = a$ for all $a \in A$.
(iii) There is a unique element $1 \in K$ satisfying $1 \cdot a = a$ for all $a \in A$.
(iv) For each $a \in K$, there is a unique element of K, denoted $-a$, such that $a + (-a) = 0$.
(v) For each $a \in K - \{0\}$, there is a unique element of K, denoted a^{-1}, such that $a \cdot a^{-1} = 1$.
(vi) Multiplication is distributive over addition; i.e., for each $a, b, c \in K$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$.
(vii) $1 \neq 0$

We write $a - b$ for $a + (-b)$, and $\frac{a}{b}$ or a/b for $a \cdot b^{-1}$.

Remark 1.7.3. This remark applies not just to fields, but to all of the many species of "structures" one encounters in mathematics.

Given a field $(F, +, \cdot)$, the set F is called the underlying set of the field. A typical "abuse of notation" is to conflate the structure with its underlying set, and just use F, rather than $(F, +, \cdot)$ to denote the field, trusting that the reader understands from context what the field operations are.

Definition 1.7.4. An ordered field $(F, +, \cdot, \leqslant)$ is a field $(F, +, \cdot)$ together with a total ordering $\leqslant$ such that, for any $a, b, c \in F$:

(i) If $a < b$, then $a + c < b + c$.
(ii) If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$.

We say that $a \in F$ is positive if $a > 0$ and negative if $a < 0$.

Definition 1.7.5. Let $(X, \leqslant)$ be any totally ordered set. Given a subset $S \subset X$, we say that an element $b \in X$ is an upper (or lower) bound of S if $b \geqslant s$ (or $b \leqslant s$) for all $s \in S$. If an upper or lower bound b of S also belong to S, we call it the maximum (or minimum) of S.

We say that an upper bound b of S is a least upper bound (or l.u.b., or supremum) if for each upper bound b' of S, we have $b \leqslant b'$. Similarly, a greatest lower bound (g.l.b. or infimum) of S is a lower-bound b of S such that $b \geqslant b'$ for all lower bounds b' of S.

(**Theorem**: any two l.u.b.'s or g.u.b.'s of a set S are equal.)

A subset $S \subset X$ is bounded above (or below) if it has an upper (or lower) bound, and bounded if it is bounded both above and below.

Finally, we say that an ordered field $(F, +, \cdot, \leqslant)$ is complete and Archimedean if each non-empty subset $S \subset F$ which is bounded above has a supremum.

Definition/Theorem 1.7.6. There exists a complete Archimedean ordered field $(\mathbb{R}, +, \cdot, \leqslant)$, which we call the field of real numbers.

The proof of this theorem, which we discussed in class, involves the construction of the field of *Dedekind cuts*. Whenever we speak of the complete Archimedean ordered field $(\mathbb{R}, +, \cdot, \leqslant)$, we will be referring to the particular field of Dedekind cuts. However, as with the natural numbers, the particular construction we used is "irrelevant", in light of the following theorem (and therefore, you should "forget" about the Dedekind cuts, and only rely on the axioms of a complete ordered field).

Theorem 1.7.7. Any two complete Archimedean ordered field are "isomorphic":

Given any two complete Archimedean ordered field $(\mathbb{R}, +, \cdot, \leqslant)$ and $(\mathbb{R}', +', \cdot', \leqslant')$, there is a (in fact, unique) bijection $f \colon \mathbb{R} \to \mathbb{R}'$ such that for all $a, b \in \mathbb{R}$ we have $f(a + b) = f(a) + f(b)$ and $f(a \cdot b) = f(a) \cdot f(b)$, and $a \leqslant b \iff f(a) \leqslant f(b)$.

Definition 1.7.8. We define a function $i \colon \mathbb{N} \to \mathbb{R}$ by recursion by setting $i(0) = 0$, and $i(n + 1) = i(n) + 1$. We call the image $i(\mathbb{N}) \subset \mathbb{R}$ of i the set of natural real numbers.

(**Theorem**: i is injective.)

Since i is injective, it induces a bijection $\mathbb{N} \cong i(\mathbb{N})$, and we will usually not bother to distinguish between a natural number and the corresponding natural real number.

We define the set of integers as $\mathbb{Z} = \{a - b \mid a, b \in i(\mathbb{N})\}$ and the set of rational numbers as $\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \backslash \{0\}\}$.

1.8. **Calamity.**

Theorem 1.8.1. There exists a set X such that $X \in X$ and $X \notin X$.

This is obviously a problem and shows that there was something wrong with our axioms, and hence we need to revise them. Specifically, we replace the Axiom of Unrestricted Comprehension with a bunch of carefully selected special cases of it (collectively called the "Axiom of Restricted Comprehension"), which suffice for all mathematical purposes, but which don't allow you to do crazy things like form the set of all sets.

We list these special cases here for the sake of completeness, but you should immediately "forget" them and go back to using unrestricted comprehension. The point is that any particular case of unrestricted comprehension you use in practice will be an instance of one of the following (or a consequence of them).

Before, we proceed, a word on notation: the notation $\{x \mid \varphi(x)\}$ was used in light of the Axiom of Unrestricted Comprehension to denote the set of all x satisfying $\varphi(x)$. Since we are abandoning this axiom, the notation $\{x \mid \varphi(x)\}$ is no longer valid in general.

However, for *certain* formulas $\varphi(x)$, there *will* still be a (by Extensionality unique) set containing just those things x satisfying $\varphi(x)$. In this case, we will still use the notation $\{x \mid \varphi(x)\}$ to denote that set.

Also, we will say "$\{x \mid \varphi(x)\}$ is a set" to mean that there is, indeed, a set containing exactly those things x satisfying $\varphi(x)$. We make similar conventions regarding the various notations we introduced for denoting sets. For example, to interpret the first point "$\varnothing$ is a set" in Axiom $2'$ below, we first expand the definition of $\varnothing$ to obtain "$\{x \mid x \neq x\}$ is a set", which therefore means "there exists a (by Extensionality unique) set containing exactly those things x such that $x \neq x$".

Finally, we note that there is one more instance of restricted comprehension (called the "axiom of replacement") which is often included but which we are omitting, since it is somewhat complicated to state, and in any case, is only needed for special set-theoretic purposes, and never needed in ordinary mathematical practice.

Axiom $2'$ ("Restricted comprehension")**.**

(i) $\varnothing$ is a set.
(ii) If a, b are any two things, then $\{a, b\}$ is a set.
(iii) If $\mathcal{A}$ is any set, all of whose elements are sets, then $\bigcup \mathcal{A}$ is a set.
(iv) If A is any set, then $\mathcal{P}A$ is a set.
(v) If A is any set and P is any property, then
$$\{x \in A \mid x \text{ satisfies } P\}$$
is a set.
(vi) ω (the set of von Neumann natural numbers) is a set.

1.9. **Indexed products, sequences, and disjoint unions.** Here are a bunch more general set-theoretic definitions.

Definition 1.9.1.

- We introduce an alternative notation for functions. Given a set J and, for each element $\alpha \in J$, an element x_α of some set X, we may denote by
$$\{x_\alpha\}_{\alpha \in J} \quad \text{or} \quad (x_\alpha)_{\alpha \in J}$$
the function $J \to X$ whose value at $\alpha \in J$ is x_α. In this case, we will often refer to the function $\{x_\alpha\}_{\alpha \in J}$ as an <u>indexed family of elements of X</u> (or just an <u>indexed family</u>), and refer to J as the corresponding <u>indexing set</u>.
- An <u>infinite sequence</u> (or just <u>sequence</u>) <u>of elements of X</u> is just a family $\{x_n\}_{n \in \mathbb{N}}$ of elements of X with indexing set $\mathbb{N}$ (i.e., a function $\mathbb{N} \to X$). We often denote an infinite sequence $\{x_n\}_{n \in \mathbb{N}}$ by $\{x_n\}_{n=0}^{\infty}$, or by
$$(x_0, x_1, x_2, \dots)$$
when the expression for the general x_n can be inferred from the first few instances. Sometimes, we may even just denote this sequence by (x_n) or even x_n.
- Given $n \in \mathbb{N}$, an <u>n-tuple</u> or <u>finite sequence of length n</u> of elements of X is just an indexed family $\{x_i\}_{i \in J}$ of elements of X in which $J = \{1, \dots, n\}$ for some $n \in \mathbb{N}_+$.

  We often denote a tuple $\{x_i\}_{i \in J}$ by $\{x_i\}_{i=1}^{n}$, or by $(x_1, \dots, x_n)$.

  Note that for $n = 2$ or $n = 3$, this conflicts with our earlier notation (a, b) or (a, b, c) for ordered pairs and triples, but this is justified in light of the obvious bijection between the set of ordered pairs/triples and the set of 2-tuples/3-tuples.
- An <u>indexed family of sets</u> is just an indexed family $\{A_\alpha\}_{\alpha \in J}$ in which each A_α is itself a set.
- Given an indexed family of sets $\{A_\alpha\}_{\alpha \in J}$, the <u>union</u> $\bigcup_{\alpha \in J} A_\alpha$ and <u>intersection</u> $\bigcap_{\alpha \in J} A_\alpha$ of this family are the sets
$$\bigcup_{\alpha \in J} A_\alpha = \bigcup \{A_\alpha \mid \alpha \in J\} \quad \text{and} \quad \bigcap_{\alpha \in J} A_\alpha = \bigcap \{A_\alpha \mid \alpha \in J\}.$$
- The <u>indexed product</u> $\prod_{x \in \alpha} A_\alpha$ of a family of sets $\{A_\alpha\}_{\alpha \in J}$ is the set of all indexed families $(x_\alpha)_{\alpha \in J}$ of elements of $\bigcup_{\alpha \in J} A_\alpha$ such that $x_\alpha \in A_\alpha$ for all α:
$$\prod_{\alpha \in J} A_\alpha = \{f : J \to \bigcup_{\alpha \in J} A_\alpha \mid f(\alpha) \in A_\alpha \text{ for all } \alpha \in J\}.$$

  If $J = \{1, \dots, n\}$ for some $n \in \mathbb{N}_+$, we may also write $A_1 \times \cdots \times A_n$ for $\prod_{i=1}^{n} A_i$. Again, when $n = 2$, this conflicts with our old definition for the product $A \times B$, but there is a canonical bijection between $A \times B$ in the old sense and $A \times B$ in the new sense.
- If $\{A_\alpha\}_{\alpha \in J}$ is a constant family – i.e., there is some set A such that $A_\alpha = A$ for all α – then we may simply write A^J rather than $\prod_{\alpha \in J} A$. Note that this is just the set of functions from J to A.

  If, moreover, $J = \{1, \dots, n\}$ for some $n \in \mathbb{N}_+$, we may write A^n for A^J.
- The <u>disjoint union</u> $\coprod_{\alpha \in J} A_\alpha$ of a family of sets $\{A_\alpha\}_{\alpha \in J}$ is the set of all pairs (α, x) where $\alpha \in J$ and $x \in A_\alpha$:
$$\coprod_{\alpha \in J} A_\alpha = \{(\alpha, x) \mid \alpha \in J \text{ and } x \in A_\alpha\} = \bigcup_{\alpha \in J} \{\alpha\} \times A_\alpha.$$
Again, if $J = \{1, \dots, n\}$, we may write $A_1 \sqcup \cdots \sqcup A_n$ for $\coprod_{i=1}^{n} A_i$. In particular, we have $A \sqcup B = \{1\} \times A \cup \{2\} \times B$.

1.10. **Finite and countable sets, and cardinality.**

Definition 1.10.1. A set A is finite if it is empty or there exists a bijection $\{1,\dots,n\} \xrightarrow{\sim} A$ for some positive $n \in \mathbb{N}$. In the former case, we say that A has cardinality 0, and otherwise that A has cardinality n.

If a set is not finite, it is infinite.

Theorem 1.10.2. *Given $m, n \in \mathbb{N}$, if a set A has cardinality m and also has cardinality n, then $m = n$.*

Definition 1.10.3. In light of the previous theorem, given a finite set A, there is a unique n such that the A has cardinality n. This n is called the cardinality of A, denoted $\#A$.

Theorem 1.10.4. *If A is finite and $B \subsetneq A$, then B is finite and $\#B < \#A$.*

Theorem 1.10.5. *$\mathbb{N}$ is infinite. More generally, for any set A, if there exists a bijection between A and a proper subset of A, then A is infinite.*

Definition 1.10.6. A set A is countably infinite if there exists a bijection $\mathbb{N} \xrightarrow{\sim} A$. It is countable if it is finite or countably infinite.

Warning: sometimes "countable" is simply used to mean "countably infinite".

If A is not countable, it is uncountable.

Definition 1.10.7. For any two sets A and B, we say that A and B are equipotent or have the same cardinality, and write $|A| = |B|$, if $A \cong B$ (i.e., if there exists a bijection between A and B).

We say that the cardinality of A is less than or equal to the cardinality of B, and write $|A| \lesssim |B|$, if there exists an injection $A \to B$.

Theorem 1.10.8 (Cantor-Schröder-Bernstein)**.** *If $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$.*

Theorem 1.10.9 (Cantor)**.** *For every set A, we have $|A| < |\mathcal{P}(A)|$. Also, $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$ (hence $\mathbb{R}$ is uncountable).*

Theorem 1.10.10. *Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of countable sets with J finite. Then $\bigcup_{\alpha \in J} A_\alpha$ and $\prod_{\alpha \in J} A_\alpha$ are also countable.*

1.11. **The axiom of choice.**

Axiom 3 (Axiom of Choice)**.** *Given any set $\mathcal{A}$ of non-empty sets, there exists a function $c\colon \mathcal{A} \to \bigcup \mathcal{A}$ with the property that $c(A) \in A$ for all $A \in \mathcal{A}$.*

Such a function c is called a choice function for $\mathcal{A}$.

Theorem 1.11.1. *Any surjective function $f\colon A \to B$ has a right-inverse (i.e., there exists a function $g\colon B \to A$ with $f \circ g = \mathrm{id}_A$).*

Theorem 1.11.2. *Let A be a set. The following statements are equivalent:*

(i) *There exists an injection $\mathbb{N} \to A$.*
(ii) *There exists a bijection between A and a proper subset of A.*
(iii) *A is infinite*

Theorem 1.11.3. *Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of countable sets with J countable. Then $\bigcup_{\alpha \in J} A_\alpha$ is also countable.*

2. Metric spaces

Definition 2.0.1. A metric d on a set X is function $d\colon X \times X \to \mathbb{R}$ satisfying, for all $x, y, z \in X$:

(i) ("Positivity") $d(x,y) \geqslant 0$; and $d(x,y) = 0 \iff x = y$
(ii) ("Symmetry") $d(x,y) = d(y,x)$
(iii) ("Triangle inequality") $d(x,z) \leqslant d(x,y) + d(y,z)$

A metric space is a pair (X,d) with X a set and d a metric on X. As mentioned in Remark 1.7.3, we will often conflate a metric space (X,d) with its underlying set X. Also, If we are considering a metric space with underlying set X and haven't given a name to the metric, we may just refer to it as d_X, or even just d.

We often call the elements of a metric space (X,d) the *points* of X.

Definition 2.0.2. Fix any $n \in \mathbb{N}_+$.

Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $a \in \mathbb{R}$, we denote by $\mathbf{x}+\mathbf{y}$, $\mathbf{x}-\mathbf{y}$, and $a\mathbf{x}$ the usual component-wise sum, difference, and scalar multiple.

We denote by $\|\mathbf{x}\|$ the number[1] $\sqrt{\sum_{i=1}^n x_i^2}$, called the (Euclidean) norm or length of $\mathbf{x}$. Note that $\|\mathbf{x}\| \geqslant 0$.

Finally, we define $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$, which we call the (Euclidean) *distance between* $\mathbf{x}$ *and* $\mathbf{y}$.

Definition/Theorem 2.0.3. d as defined above is a metric on $\mathbb{R}^n$, called the Euclidean metric on $\mathbb{R}^n$.

Definition 2.0.4. If (X,d) is a metric space and $A \subset X$ is a subset, then the restriction $d|_{A\times A}$ is called the induced metric or subspace metric on A.

(**Theorem**: it is a metric.)

Definition/Theorem 2.0.5. For any $n \in \mathbb{N}_+$, the function $d_{\mathrm{Tc}}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $d_{\mathrm{Tc}}(\mathbf{x},\mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$ is a metric on $\mathbb{R}^n$, called the taxicab metric.

Definition/Theorem 2.0.6. For any set X, the function $d\colon X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric on X, called the discrete metric.

Definition/Theorem 2.0.7. Given metric spaces (X_1,d_1) and (X_2,d_2), the function $d\colon (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$ defined by $d\big((x_1,x_2),(y_2,y_2)\big) = d_1(x_1,y_1) + d_2(x_2,y_2)$ is a metric, called the standard product metric.

2.1. Open and closed sets.

Let (X,d) be a metric space until further notice.

Definition 2.1.1. Given a point $x \in X$ and a real number $r \geqslant 0$, we define the open ball of radius r at x to be the subset $\mathrm{B}_r(x) = \{y \in X \mid d(x,y) < r\}$, and the closed ball of radius r at x to be the subset $\overline{\mathrm{B}}_r(x) = \{x \in X \mid d(r,x) \leqslant r\}$. Note that $\mathrm{B}_0(x) = \varnothing$ and $\overline{\mathrm{B}}_0(x) = \{x\}$.

[1] We have not formally introduced the summation notation; it is defined by an obvious recursion: for each $n \in \mathbb{N}_+$, we define a function $\sum_{i=1}^n\colon \mathbb{R}^n \to R$ recursively in n by setting $\sum_{i=1}^1 a_i = a_i$ and $\sum_{i=1}^{n+1} a_i = \sum_{i=1}^n a_i + a_n$. One proves various familiar properties of the sum by induction, such as "distributivity" $a\sum_{i=1}^n b_i = \sum_{i=1}^n ab_i$. In the future, we will simply assume such familiar properties without further discussion.

The square root $\sqrt{x}$ of a non-negative real number x was defined in the homework. It is the unique $y \in \mathbb{R}_{\geqslant 0}$ with $y^2 = x$. Also note that $\sum_{i=1}^n x_i^2$ is positive, being a sum of squares.

When we need to emphasize the dependence of $\mathrm{B}_r(x)$ and $\overline{\mathrm{B}}_r(x)$ on the metric d, we may write $\mathrm{B}_r(x; d)$ and $\overline{\mathrm{B}}_r(x; d)$.

A subset $U \subset X$ is open if for each $x \in U$ there is some $r > 0$ with $\mathrm{B}_r(x) \subset U$. A subset is closed if its complement is open.

If $U \subset X$ is open and contains $x \in X$, we say that U is an open neighborhood of X.

Theorem 2.1.2.

(i) X and $\varnothing$ are open.
(ii) If $\mathcal{U} \subset \mathcal{P}(X)$ is any set of open sets, then $\bigcup \mathcal{U}$ is open.
(iii) If $\mathcal{U} \subset \mathcal{P}(X)$ is any (non-empty) *finite* set of open sets, then $\bigcap \mathcal{U}$ is open.

Remark 2.1.3. Here is an extremely trivial remark:

The reason that we require $\mathcal{U}$ to be non-empty in (iii) is that we only defined the intersection $\bigcap \mathcal{A}$ when $\mathcal{A}$ is non-empty.

However, when $\mathcal{A} \subset \mathcal{P}(Y)$ is a set of subsets of some given set Y (as is the case here), it is natural to define $\bigcap \mathcal{A}$ as

$$\bigcap \mathcal{A} = \{y \in Y \mid y \in A \text{ for all } A \in \mathcal{A}\}$$

rather than

$$\bigcap \mathcal{A} = \{y \mid y \in A \text{ for all } A \in \mathcal{A}\}.$$

This new definition gives the exact same result whenever $\mathcal{A} \neq \varnothing$, but when $\mathcal{A} = \varnothing$, it gives $\bigcap \varnothing = Y$.

Hence, if we take this new definition, then we do not need to demand "non-empty" in (iii).

Theorem 2.1.4. For any $x \in X$ and $r \geqslant 0$, the open ball $\mathrm{B}_r(x)$ is open and the closed ball $\overline{\mathrm{B}}_r(x)$ is closed.

2.2. **Continuous maps and homeomorphisms.** Let (X, d_X) and (Y, d_Y) be metric spaces.

Definition 2.2.1. Let $f : X \to Y$ be a function. Given $x \in X$, the function f is continuous at x if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$ for all $x' \in X$ (or, in other words, $f(\mathrm{B}_\delta(x)) \subset \mathrm{B}_\varepsilon(f(x))$). The function f is continuous if it is continuous at x for every $x \in X$.

The function f is a homeomorphism if it is continuous and bijective, and its inverse $f^{-1} : Y \to X$ is also continuous.

In this case, we say that (X, d_X) and (Y, d_Y) are homeomorphic.

Theorem 2.2.2. A function $f : X \to Y$ is continuous if and only if, for each open $U \subset Y$, the preimage $f^{-1}(U)$ is open in X.

3. Topological spaces

Definition 3.0.1. A topology on a set X is a set $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X satisfying the following:

(i) X and $\varnothing$ are in $\mathcal{T}$
(ii) If $\mathcal{U} \subset \mathcal{T}$ is any subset of $\mathcal{T}$, then $\bigcup \mathcal{U} \in \mathcal{T}$.
(iii) If $\mathcal{U} \subset \mathcal{T}$ is any (non-empty) *finite* subset of $\sqcup$, then $\bigcap \mathcal{U} \in \mathcal{T}$. (Equivalently (by induction), whenever $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.)

We call the elements of $\mathcal{T}$ the open subsets of X, and we call a subset closed if its complement is open.

A topological space (or just space) is a pair $(X, \mathcal{T})$ with X a set and $\mathcal{T}$ a topology on X.

(As usual – see Remark 1.7.3 and Definition 2.0.1 – we may sometimes use X to refer to the topological space $(X, \mathcal{T})$. Also, if we are considering a topological space with underlying X and have not given a name to the topology, we must just refer to it as $\mathcal{T}_X$, or even just $\mathcal{T}$.)

Definition/Theorem 3.0.2. If (X, d) is a metric space, then the set $\mathcal{T}$ of all open subsets of X is a topology, called the *metric topology* on (X, d).

A topological space $(X, \mathcal{T})$ is metrizable if $\mathcal{T}$ is the metric topology for some metric d on X.

Definition 3.0.3. Given a set X, the set of all subsets of X is called the discrete topology, and the set $\{\varnothing, X\} \subset X$ is called the indiscrete topology.

(**Theorem**: These are indeed both topologies.)

Definition 3.0.4. Let $\mathcal{T}, \mathcal{T}' \subset \mathcal{P}(X)$ be topologies on a set X. If $\mathcal{T} \subset \mathcal{T}'$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ and that $\mathcal{T}$ is coarser than $\mathcal{T}'$

(We might also simply say – in light of "$\subset$" being a partial order on $\mathcal{P}(X)$ – that $\mathcal{T}$ is smaller than $\mathcal{T}'$.)

Thus, for example, the discrete and indiscrete topologies are respectively the finest and coarsest topologies on X.

Definition 3.0.5. Let $(X, \mathcal{T}_X)$ and $(T, \mathcal{T}_Y)$ be topological spaces.

A map $f: X \to Y$ is continuous if for each open $U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$.

We say that f is a homeomorphism or isomorphism of topological spaces, and write $f: X \xrightarrow{\sim} Y$ if f is continuous and bijective, and its inverse $f^{-1}: Y \to X$ is also continuous.

In this case, we say that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic and write $(X, \mathcal{T}_X) \cong (Y, \mathcal{T}_Y)$ (or just $X \cong Y$).

(**Theorem**: f is continuous only if $f^{-1}(V) \subset X$ is closed for every closed subset $V \subset Y$.)

Definition/Theorem 3.0.6. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ a subset. Then the set

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\} = \{U \mid U = V \cap A \text{ for some } V \in \mathcal{T}\} \subset \mathcal{P}(A)$$

is a topology on A, called the subspace topology.

A subspace of $(X, \mathcal{T})$ is a topological space given by some subset $A \subset X$ endowed with subspace topology.

Definition 3.0.7. Given a subset $A \subset X$ of a topological space X, we say that a subset $S \subset A$ is open in A if it is open in the subspace topology on A (i.e., if $U = A \cap V$ for some open $V \subset X$), and closed in A if it is closed in the subspace topology on A (i.e., if $A - S$ is open in A).

Theorem 3.0.8. If $A \subset X$ is an open subset of a topological space X, then a subset $S \subset A$ is open in X if and only if it is open in X. If A is closed, then $S \subset A$ is closed in A if and only if it is closed in X.

Theorem 3.0.9. Let X be a metric space and $A \subset X$ a subset. Then the subspace topology on A induced by the metric topology on X is the same as the metric topology on A coming from the subspace metric on $A \subset X$.

Remark 3.0.10. Note that in the above theorem, we just wrote "Let X be a metric space" rather than "Let (X, d) be a metric space", since we didn't need to make any explicit reference to the metric d in the theorem. This is in accordance with Remark 1.7.3, and we will continue to do this when it is convenient.

3.1. Bases and subbases, the product topology, and continuity at a point.

Definition 3.1.1. Given a topological space $(X, \mathcal{T})$, we say that a set $\mathcal{B} \subset \mathcal{P}(\mathcal{T})$ is a basis for $\mathcal{T}$ if every open subset $U \in \mathcal{T}$ is a union of elements of $\mathcal{B}$ – or equivalently, for each open $U \subset X$ and each $x \in U$, there is some $B \in \mathcal{B}$ with $x \in B \subset U$.

(**Theorem**: For a metric space (X, d), the open balls are a basis for the metric topology.)

When we have fixed a basis $\mathcal{B}$, we call the elements $U \in \mathcal{B}$ *basic open sets*.

Definition/Theorem 3.1.2. Let X be a set and $\mathcal{B} \subset \mathcal{P}(X)$ be a collection of subsets. Then $\mathcal{B}$ is a basis for some topology on X if and only if:

(i) Each $x \in X$ is contained in some $B \in \mathcal{B}$. (Equivalently, $X = \bigcup \mathcal{B}$.)
(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$. (Equivalently, $B_1 \cap B_2$ is a union of elements of $\mathcal{B}$.)

In this case, there is a unique topology $\mathcal{T}$ with basis $\mathcal{B}$ – called the topology generated by $\mathcal{B}$ – and consists of all unions of elements of $\mathcal{B}$ (equivalently: a set $U \subset X$ is in $\mathcal{T}$ if and only if for each $x \in U$, there is some $B \in \mathcal{B}$ with $x \in B \subset U$.)

Definition/Theorem 3.1.3. Given topological spaces X and Y, the set $\mathcal{B} \subset \mathcal{P}(X \times Y)$ of subsets of $X \times Y$ given by

$$\mathcal{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in Y}\}$$

(satisfies the condition of Theorem 3.1.2 and hence) is the basis for a topology on $X \times Y$. This is called the product topology.

Theorem 3.1.4. Let X and Y be metric spaces. Then the metric topology coming from the standard product metric on $X \times Y$ is the same as the product topology coming from the metric topologies on X and Y.

Definition 3.1.5. Given topological spaces X and Y and a point $x \in X$, we say that a function $f\colon X \to Y$ is continuous at x if for each open neighborhood $V \subset Y$ of y, there is an open neighborhood $U \subset X$ of x such that $f(U) \subset V$.

Theorem 3.1.6. Let $f\colon X \to Y$ be a function between topological spaces X and Y.

(i) f is continuous if and only if it is continuous at x for each $x \in X$.
(ii) Given bases $\mathcal{B}$ for X and $\mathcal{B}'$ for Y, we have that f is continuous at $x \in X$ if and only if, for each $B' \in \mathcal{B}'$ containing y, there is some $B \in \mathcal{B}$ containing x such that $f(B) \subset B'$.
(iii) In particular, if X and Y are metrizable, then f is continuous at $x \in X$ if and only if it continuous at x in the sense of Definition 2.2.1.

Lemma 3.1.7. For any non-empty family $\{\mathcal{T}_\alpha\}_{\alpha \in J}$ of topologies on a set X, the intersection $\bigcap_{\alpha \in J} \mathcal{T}_\alpha$ is also a topology.

Definition/Theorem 3.1.8. Given a topological space $(X, \mathcal{T})$, a subset $\mathcal{B} \subset \mathcal{T}$ is a subbasis for $\mathcal{T}$ if $\bigcup \mathcal{B} = X$, and $\mathcal{T}$ is the coarsest topology on X with $\mathcal{B} \subset \mathcal{T}$.

Given any set $\mathcal{B} \subset \mathcal{P}(X)$ of subsets of a set X, there is a unique topology $\mathcal{T}$ for which $\mathcal{B}$ is a subbasis, called the topology generated by $\mathcal{B}$. It is the intersection of all topologies $\mathcal{T}'$ containing $\mathcal{B}$, and consists precisely of all unions of finite intersections of elements of $\mathcal{B}$.

(Note that the set of all finite intersections of elements in $\mathcal{B}$ forms a basis for $\mathcal{T}$.)

Definition 3.1.9. Given a product $\prod_{\alpha \in J} X_\alpha$ of an indexed family $\{X_\alpha\}_{\alpha \in J}$ of sets, the associated product projections are the maps $\pi_\beta\colon \prod_{\alpha \in J} X_\alpha \to X_\beta$ for each $\beta \in J$, defined by $\pi_\beta\Big((x_\alpha)_{\alpha \in J}\Big) = x_\beta$.

Definition 3.1.10. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces.

We defined the product topology on $\prod_{\alpha \in J} X_\alpha$ to be the coarsest topology for which each product projection $\pi_\beta \colon \prod_{\alpha \in J} X_\alpha \to X_\beta$ is continuous.

In other words, the product topology has as a subbasis all sets of the form

$$\pi_\beta^{-1}(U) = \{(x_\alpha)_{\alpha \in J} \mid x_\beta \in U\}$$

with $\beta \in J$ and $U \subset X_\beta$ open.

It has as a basis all sets of the form

$$\{(x_\alpha)_{\alpha \in J} \mid x_{\alpha_1} \in U_1, \ldots, x_{\alpha_n} \in U_n\}$$

with $\alpha_1, \ldots, \alpha_n \in J$ and $U_i \subset X_{\alpha_i}$ open for each $i = 1, \ldots, n$.

Remark 3.1.11. Note that in the case $J = \{1, 2\}$, this definition specializes to the product topology on $X_1 \times X_2$, as defined in Definition 3.1.3.

Theorem 3.1.12. The metric topology on $\mathbb{R}^n = \prod_{i=1}^n \mathbb{R}$ agrees with the product topology (where each factor $\mathbb{R}$ is given the metric topology).

3.2. Constructing continuous functions.

Theorem 3.2.1. Let X and Y be topological spaces, and let $A \subset X$ and $B \subset Y$ be subspaces.

(i) If $f \colon X \to Y$ is continuous, then so is the restriction $f|_A \colon A \to Y$.

(ii) Given a function $f \colon X \to Y$ with $f(X) \subset B$, then f is continuous if and only if the function $f \colon X \to B$ obtained by shrinking the codomain of f is continuous.

Theorem 3.2.2. The maps $+, \cdot \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the map $- \colon \mathbb{R} \to \mathbb{R}$, and the map ${}^{-1} \colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$ are continuous. Also, $\sqrt{\ } \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is continuous.

Here is a useful lemma in the proof:

Lemma 3.2.3. Let $f \colon X \to Y$ be a map of metric spaces. If, for each $x \in X$, there is a constant $C > 0$ such that $d_Y(f(x), f(x')) \leqslant C d_X(x, x')$ for all $x' \in X$, then f is continuous.

More generally, if for each $x \in X$, there is some $\delta > 0$ and some $C > 0$ such that $d_Y(f(x), f(x')) \leqslant C d_X(x, x')$ for all $x' \in \mathrm{B}_\delta(x)$, then f is continuous.

Theorem 3.2.4. Let X, Y, Z be topological spaces.

(i) If $f \colon X \to Y$ is a constant function (i.e., there is some $y_0 \in Y$ such that $f(X) = \{y_0\}$) then f is continuous.

(ii) Given a subspace $A \subset X$, the inclusion function $i \colon A \to X$ (given by $i(a) = a$) is continuous. In particular, id_X is continuous.

(iii) ("Topological spaces and continuous maps form a category") id_X is continuous, and if $f \colon X \to Y$ and $g \colon Y \to Z$ are continuous, then so if $g \circ f \colon X \to Z$.

Theorem 3.2.5. Let X be a topological space and let $\{Y_\alpha\}_{\alpha \in J}$ be a family of topological spaces.

A map $f \colon X \to \prod_{\alpha \in J} Y_\alpha$ into the product is continuous if and only if the composite $\pi_\beta \circ f \colon X \to Y_\beta$ of f with each projection map $\pi_\beta \colon \prod_{\alpha \in J} Y_\alpha \to Y_\beta$ is continuous.

Corollary 3.2.6.

- If X is a topological space and $f, g \colon X \to \mathbb{R}$ are continuous functions, then so is $f + g \colon X \to \mathbb{R}$ defined by $(f+g)(x) = f(x) + g(x)$ and $f \cdot g$ defined similarly, and also $\frac{f}{g}$, which is defined on the subset $\{x \in X \mid g(x) \neq 0\}$.

- Any polynomial function $p\colon \mathbb{R} \to \mathbb{R}$, i.e., one of the form $p(x) = \sum_{i=0}^{n} a_i x^i$ is continuous, as is any rational function, i.e., one of the form $f(x) = \frac{p(x)}{q(x)}$, where $p, q\colon \mathbb{R} \to \mathbb{R}$ are polynomial functions (and which is defined on the set $\{x \in \mathbb{R} \mid q(x) \neq 0\}$ is continuous.
- Similarly, polynomial functions in several variables, such as any function $p\colon \mathbb{R}^2 \to \mathbb{R}$ of the form $p(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} x^i y^j$, are continuous.
- The Euclidean norm $\|-\|\colon \mathbb{R}^n \to \mathbb{R}$ is continuous.

Theorem 3.2.7. Let X and Y be topological spaces and let $f\colon X \to Y$ be a map.

(i) ("Continuity is a local condition") Given a collection $\mathcal{U} \subset \mathcal{P}(X)$ of open sets such that $\bigcup \mathcal{U} = X$, then f is continuous if and only if $f|_U\colon U \to Y$ is continuous for each $U \in \mathcal{U}$ (equivalently, each $x \in X$ has an open neighborhood U such that $f|_U$ is continuous).
(ii) ("Pasting lemma") If $A, B \subset X$ are closed and $X = A \cup B$, and if the restrictions $f|_A\colon A \to Y$ and $f|_B\colon B \to Y$ are continuous, then so is f.

Corollary 3.2.8. The absolute value function $|-|\colon \mathbb{R} \to \mathbb{R}$ is continuous, and so is the taxicab norm $\|-\|_{\mathrm{Tc}}\colon \mathbb{R}^n \to \mathbb{R}$.

3.3. **Closures, interiors, and sequences.**

Definition 3.3.1. Given a subset A of a topological space X, the interior of A, denote $\mathrm{Int}A$, is the union all of all open sets contained in A (or equivalently, the largest open set contained in A), and the closure of A, denoted $\mathrm{Cl}A$ or $\bar{A}$, is the intersection of all closed sets containing A (or equivalently the smallest closed set containing A).

The boundary ∂A of A is the set $\partial A = \bar{A} - \mathrm{Int}(A)$.

Note that $\mathrm{Int}A \subset A \subset \bar{A}$, and the first or second inclusion is an equality iff A is open or closed, respectively.

Theorem 3.3.2. A subset A of a topological space X is closed if and only if $\partial A \subset A$, and is open if and only if $\partial A \cap A = \varnothing$.

Theorem 3.3.3. Given a subset A of a topological space X and a point $x \in X$:

(i) $x \in \mathrm{Int}A$ if and only if there is some open neighborhood of X contained in A
(ii) $x \in \bar{A}$ if and only if each open neighborhood of x intersects A
(iii) $x \in \partial A$ if and only if each neighborhood of x intersects both A and $X - A$.

Definition 3.3.4. Given a sequence (x_n) in a set X and a subset $U \subset X$, we say that (x_n) is eventually in U if there is some $N \in \mathbb{N}$ such that $x_n \in U$ for $n \geqslant N$.

If X is a topological space, we say that (x_n) converges to a point $x \in X$, or that x is a limit of (x_n), and write $x_n \xrightarrow{n\to\infty} x$, if for each open neighborhood U of x, (x_n) is eventually in U.

Theorem 3.3.5.

(i) If X is a metric space, then $x_n \xrightarrow{n\to\infty} x$ if and only if for each $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $d(x, x_n) < \varepsilon$ for all $n \geqslant N$.
(ii) More generally, if $\mathcal{B}$ is a basis for the topological space X, then in the definition of $x_n \xrightarrow{n\to\infty} x$, it suffices to consider open neighborhood U of x with $U \in \mathcal{B}$.

Theorem 3.3.6. Let X be a topological space, let $A \subset X$ a subset, and let $y \in X$. Then the following three implications hold. If X is metrizable, then the reverse implications hold as well.

There is a sequence (y_n) in A with $y_n \xrightarrow{n\to\infty} y$.	$\Longrightarrow$ $\underset{X \text{metrizable}}{\Longleftarrow}$	$y \in \bar{A}$
A is closed	$\Longrightarrow$ $\underset{X \text{metrizable}}{\Longleftarrow}$	For any sequence (x_n) in A, if (x_n) coverges to some $x \in X$, we have $x \in A$ (i.e., A is "closed under limits").
A is open	$\Longrightarrow$ $\underset{X \text{metrizable}}{\Longleftarrow}$	For any sequence (x_n) in X, if (x_n) converges to some $x \in A$, then (x_n) is eventually in A.

Theorem 3.3.7. Let $f\colon X \to Y$ be a map between topological spaces. Then the following implication holds, and the reverse implication holds if X is metrizable.

f is continuous	$\Longrightarrow$ $\underset{X \text{metrizable}}{\Longleftarrow}$	For each sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to some $x \in X$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ in Y converges to $f(x)$; i.e., $x_n \xrightarrow{n\to\infty} x \implies f(x_n) \xrightarrow{n\to\infty} f(x)$.

Definition 3.3.8. A topological space X is Hausdorff or T_2 if any two points have disjoint open neighborhoods; i.e., for any $x, y \in X$, there are open sets $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$.

Theorem 3.3.9. Metric spaces are Hausdorff.

Theorem 3.3.10. If a topological space X is Hausdorff, then any sequence in X has at most one limit.

Definition 3.3.11. Let X be a topological space.

- Two points $x, y \in X$ are topologically distinguishable if there is an open set U such that $x \in U \not\ni y$ or $x \notin U \ni y$.
- X is T_0 if any two points are topologically distinguishable.
- X is T_1 if for any two points $x, y \in X$, there is an open set U such that $x \in U \not\ni y$. (Equivalently, each singleton set $\{x\}$ in X is closed.)
- X is T_3 or regular Hausdorff if "points and closed sets have disjoint open neighborhoods": it is T_1 and, given any point $x \in X$ and closed set $C \subset X$, there are open sets $U \ni x$ and $V \supset C$ such that $U \cap V = \varnothing$.
- X is T_4 or normal Hausdorff if "any two closed sets have disjoint open neighborhoods": it is T_1 and, given any two closed sets $B, C \subset X$, there are open sets $U \supset B$ and $V \supset C$ such that $U \cap V = \varnothing$.

Note that $\mathrm{T}_4 \Rightarrow \mathrm{T}_3 \Rightarrow \mathrm{T}_2 \Rightarrow \mathrm{T}_1 \Rightarrow \mathrm{T}_0$.

Theorem 3.3.12. Every metric space is normal Hausdorff.

Definition 3.3.13. Let X be a topological space and $A \subset X$ a subset. A point $x \in X$ is a limit point if it satisfies either of the following equivalent conditions:

(i) every open neighborhood of x intersects $A - \{x\}$
(ii) $x \in \overline{A - \{x\}}$

The set of limit points is denoted A'. Note that $\bar{A} = A \cup A'$.

Note also that if (a_n) is a sequence in $A - \{x\}$ and $a_n \xrightarrow{n\to\infty} x$, then $x \in A'$. If X is metrizable, the converse is true as well.

3.4. **Connectedness.**

Definition 3.4.1. A topological space X is connected if the only open partition $\{U, V\}$ (i.e., $U, V \subset X$ are open and $U = V - X$) of $\{U, V\} = \{\varnothing, X\}$.

Remark 3.4.2. Here, we are using the word "partition" in a slightly different way than we did in Definition 1.4.3, since we are allowing the empty set to be an element of a partition. If we wanted to be strictly correct, we should say that X is connected if the only open partition of X is $\{X\}$. However, going forward, we will not worry so much about this distinction.

Definition 3.4.3. A subset of a topological space is clopen if it is both open and closed.

Theorem 3.4.4. A topological space X is connected iff and only if the only clopen subsets of X are X and $\varnothing$.

Definition 3.4.5. A connected component of a topological space X is a maximal connected subset of X; i.e., a connected subset $U \subset X$ such that if $U \subsetneq V \subset X$, then V is not connected.

Theorem 3.4.6. For any topological space X, the set of all connected components of X form a partition of X.

The proof uses the following two lemmas.

Lemma 3.4.7. If $A \subset X$ is a connected subset of a space X and $\{U, V\}$ is an open partition of X, then $A \subset U$ or $A \subset V$.

Lemma 3.4.8. If $\mathcal{A} \subset \mathcal{P}(X)$ is a (non-empty) collection of connected subsets of a topological space X and $\bigcap \mathcal{A}$ is non-empty, then the union $Y = \bigcup \mathcal{A}$ is connected.

Definition 3.4.9. Let $(X, \leqslant)$ be a totally ordered set. A subset $A \subset X$ is convex if for any $a, b \in A$ with $a < b$, we have $[a, b] \subset A$, i.e., $c \in A$ for all $a < c < b$.

In $\mathbb{R}$ the convex subsets are precisely $\mathbb{R}$ itself, the open, half-open, and closed intervals (a, b), $(a, b]$, $[a, b)$, and $[a, b]$, and the open and closed rays (a, ∞), $(-\infty, b)$, $[a, \infty)$, and $(-\infty, b]$.

Theorem 3.4.10. A subset $A \subset \mathbb{R}$ is connected if and only if it is convex. In particular, $\mathbb{R}$ is connected, as is any interval or ray.

Theorem 3.4.11 (Intermediate value theorem)**.** Let X be a connected topological space and let $f \colon X \to \mathbb{R}$ be a continuous map. Then for any $a, b \in X$ and $r \in \mathbb{R}$ between $f(a)$ and $f(b)$ (i.e., $r \in \big(f(a), f(b)\big)$ or $r \in \big(f(b), f(a)\big)$), there is some $c \in X$ with $f(c) = r$.

(The classical intermediate value theorem is the special case $X = [a, b]$, which is connected by Theorem 3.4.10.)

The proof uses the following lemma.

Lemma 3.4.12. Let X be a connected topological space and $f\colon X \to Y$ a continuous map. Then the image $f(X)$ is also connected.

Theorem 3.4.13. Any finite product of connected topological spaces is connected.

In particular, $\mathbb{R}^n$ is connected.

Definition 3.4.14. Given points $a, b \in X$ in a topological space X, a path from a to b is a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$.

A topological space is *path-connected* if there is a path joining any two points in X.

Theorem 3.4.15. A path-connected topological space is connected.

Theorem 3.4.16. The spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

Definition/Theorem 3.4.17. For a space X and points $x, y \in X$, define $x \sim y$ to mean that there exists a path from x to y.

Then $\sim$ is an equivalence relation. The equivalence classes are called the path components of X.

Definition 3.4.18. A space X is locally path connected if for each point x and each neighborhood U of x, there is a smaller neighborhood $V \subset U$ of x which is path-connected.

Theorem 3.4.19. Let X be a locally path connected space. Then the path components of X are the same as the connected components.

In particular, (if X is locally path connected, then) X is connected if and only if it is path connected.

3.5. **Compactness.**

Definition 3.5.1. A collection $\mathcal{U} \subset \mathcal{P}(X)$ of subsets of a space X is said to cover X, or to be a covering (or a cover) of X, if $\bigcup \mathcal{U} = X$. It is an open cover if each $U \in \mathcal{U}$ is open.

A space X is compact if every open covering $\mathcal{U} \subset \mathcal{P}(X)$ contains a finite subcover, i.e., there is a finite subset $\mathcal{V} \subset \mathcal{U}$ which also covers X.

Remark 3.5.2. Sometimes, it is more convenient to define a covering of X as an indexed family $\{U_\alpha\}_{\alpha \in J}$ of subsets of X such that $\bigcup_{\alpha \in J} U_\alpha = X$ (rather than as a set $\mathcal{U} \subset \mathcal{P}(X)$ of subsets of X).

With this notion of covering, X is compact if and only if for each such covering $\{U_\alpha\}_{\alpha \in J}$, there is a finite subset $J' \subset J$ such that $\{U_\alpha\}_{\alpha \in J'}$ is still a covering (of course, the two definitions of compactness are equivalent).

Theorem 3.5.3. If X is a compact space, then for any continuous map $f\colon X \to Y$, the image $f(X)$ is compact.

The proof uses the following lemma.

Lemma 3.5.4. Let $A \subset X$ be a subspace of a space X. Then A is compact if and only if for each covering of A by open sets in X admits a finite subcover; i.e., given a collection $\mathcal{U} \subset \mathcal{P}(X)$ of open sets in X such that $A \subset \bigcup \mathcal{U}$, there is a finite subset $\mathcal{V} \subset \mathcal{U}$ such that $A \subset \bigcup \mathcal{V}$.

Theorem 3.5.5. If X is a compact space, then any continuous map $f\colon X \to \mathbb{R}$ attains a maximal and minimal value, i.e., there exist $x, z \in X$ such that $f(x) \leqslant f(y) \leqslant f(z)$ for all $y \in X$.

Theorem 3.5.6. Any closed interval $[a, b] \subset \mathbb{R}$ is compact.

Theorem 3.5.7. A finite product of compact spaces is compact.

The proof uses the following two lemma.

Lemma 3.5.8 (Tube lemma)**.** Let X and Y be spaces, with Y compact. If N is an open subset of $X \times Y$ such that $\{x_0\} \times Y \subset N$ for some $x_0 \in X$, then $W \times X \subset Y$ for some open neighborhood $W \subset X$ of x_0.

Theorem 3.5.9. A subset of $\mathbb{R}^n$ is compact if and only if it closed and *bounded* (meaning that it is contained in some ball $\mathrm{B}_R(0)$).

The proof uses the following two lemmas.

Lemma 3.5.10. Any closed subset of a compact space is compact.

Lemma 3.5.11. Any compact subset of a Hausdorff space is closed.

This lemma uses the following lemma:

Lemma 3.5.12. If $K \subset X$ is a compact subspace of a Hausdorff space X and $x \in X - K$, then there are disjoint open sets $U, V \subset X$ with $K \subset X$ and $x \in V$

Remark 3.5.13. You may want to compare this the definition of "T_3/Hausdorff regular" (Definition 3.3.11). In particular, it follows from the previous lemmas that compact Hausdorff spaces are T_3.

Theorem 3.5.14. If $f\colon X \to Y$ is a continuous bijection where X is compact and Y is Hausdorff, then f is a homeomorphism.

Definition 3.5.15. A topological space is <u>limit point compact</u> if each infinite set has a limit point.

A space X is <u>sequentially compact</u> if each sequence has a convergence subsequence, i.e., for any sequence $(x_n)_{n\in\mathbb{N}}$ in X, there exists an increasing sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers such that the sequence X_{n_i} converges.

Theorem 3.5.16. Any compact space is limit point compact and any sequentially compact space is limit point compact.

Theorem 3.5.17. If X is a metrizable space, then the following are equivalent:

(1) X is compact
(2) X is limit-point compact
(3) X is sequentially compact

The proof of (3)⇒(1) uses the following two lemmas.

Lemma 3.5.18 (Lebesgue number lemma)**.** Let X be a sequentially compact metric space and let $\mathcal{U}$ be an open covering of X. Then there exists an $\delta > 0$ such that $\mathrm{B}_\delta(x)$ is contained in some $U \in \mathcal{U}$ for every $x \in X$.

Lemma 3.5.19 (ε-net lemma)**.** Let X be a sequentially compact metric space and let $\mathcal{U}$ be an open covering of X. Then for every $\varepsilon > 0$, there exists a covering of X by finitely many open balls $\mathrm{B}_\varepsilon(x)$ (such a covering is called an "ε-net").

The Lebesgue lemma has the following nice corollary about maps between metric spaces (note that this theorem really uses the metric; it is a statement about *metric* spaces, not *metrizable* spaces).

Theorem 3.5.20. If (X, d_X) and (Y, d_Y) are metric spaces, and X is compact, then any continuous map $f\colon X \to Y$ is *uniformly continuous*, meaning that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x_1, x_2 \in X$, if $d(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$.

3.6. **The quotient topology.**

Definition 3.6.1. Let $\sim \subset X \times X$ be an equivalence relation on a topological space X The quotient map associated to $\sim$ is the map $q_X \colon X \to X/\!\sim$ defined by $q_X(x) = [x] \in X/\!\sim$ for $x \in X$.

We define the quotient topology on $X/\sim$ by declaring that a subset $U \subset X/\sim$ is open if and only if $q_X^{-1}(X)$ is open (equivalently: $A \subset X/\sim$ is closed if and only if $q_X^{-1}(A)$ is closed); equivalently, it is the finest topology on $X/\sim$ for which q_X is continuous.

Theorem 3.6.2.

(i) Let X and Y be sets, let $\sim$ be an equivalence relation on a X, and let $f \colon X \to Y$ be a function.

Denote as before by $q_X \colon X \to X/\!\sim$ the quotient map.

Then there exists a map $\bar{f} \colon X \to Y$ with $\bar{f} \circ q_X = f$ if and only if f respects the equivalence relation $\sim$, i.e. f satisfies $f(x) = f(x')$ whenever $x \sim x'$.

Moreover, such an $\bar{f}$ is uniquely determined and is given by $\bar{f}([x]) = f(x)$ for any $x \in X$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle q_X}\downarrow & \nearrow_{\bar{f}} & \\ X/\!\sim & & \end{array}$$

(ii) Let $f \colon X \to Y$ be a map of topological spaces and let $\sim \subset X \times X$ be an equivalence relation on X. Suppose f respects the equivalence relation $\sim$.

Then f is continuous if and only if the induced map $\bar{f} \colon X/\!\sim \to Y$ is continuous.

Remark 3.6.3. Consider the relation on $R \subset [0,1] \times [0,1]$ given by $(0,t)R(1,t)$ for $t \in [0,1]$ and $(s,0) \sim (s,1)$ for $s \in [0,1]$. We might want to speak of the quotient by R – namely, the space obtained from $[0,1] \times [0,1]$ by glueing opposite sides – but we cannot, since R is not an equivalence relation. In such cases, we can use the equivalence relation generated by R.

In general, given a relation $R \subset X \times X$ on a set X, the equivalence relation $\sim \subset X \times X$ generated by R is characterized by the following equivalent conditions:

(1) $\sim_R$ is the smallest equivalence relation $\sim \subset X \times X$ on X containing R (i.e., with $R \subset \sim$).
(2) $\sim_R$ is the intersection of all equivalence relations on X containing R.
(3) Explicitly, we have $x \sim_R y$ if and only if there exists $n \geqslant 1$ and $x_1, \ldots, x_n \in X$ with $x_1 R x_2 R \cdots R x_n$ with $(x,y) = (x_1, x_n)$ or $(y,x) = (x_1, x_n)$.

Definition 3.6.4. Let $f \colon X \to Y$ be a map of sets. We define the kernel relation of X to be the equivalence relation $\sim_f \subset X \times X$ given by $x \sim_f y \iff f(x) = f(y)$.

Remark 3.6.5. Let $f \colon X \to Y$ be a surjective map between sets and consider the kernel relation $\sim_f$ on X. It is clear that f respects the equivalence relation $\sim_f$.

Hence, by Theorem 3.6.2 (i), we have an induced map $\bar{f} \colon X/\!\sim_f \to Y$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle q_X}\downarrow & \nearrow_{\bar{f}} & \\ X/\!\sim_f & & \end{array}$$

It is easy to see that (assuming f is surjective), the induced map $\bar{f}$ is always a bijection.

(In other words, "every surjective map is a quotient map, up to a bijection".)

(More generally, even if f is not surjective, one can say that $\bar{f}$ is always a bijection $\bar{f}\colon X/{\sim_f} \to f(X)$ onto the image of f; this is an analog of the "first isomorphism theorem" form algebra.)

Definition/Theorem 3.6.6. Let $f\colon X \to Y$ be a surjective map of topological spaces. If either of the following equivalent conditions hold, we say that f is a quotient map:

(i) A subset $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open (equivalently: $A \subset Y$ is closed if and only if $f^{-1}(U) \subset X$ is closed).
(ii) The induced map $\bar{f}\colon X/{\sim_f} \to Y$ is a homeomorphism.

Remark 3.6.7. Note that in Definition 3.6.1, we talk about *the* quotient map associated to an equivalence relation on a set, whereas in Definition/Theorem 3.6.6, we talk more generally about an arbitrary surjective map being *a* quotient map.

Of course, *the* quotient map associated to an equivalence relation is a particular case of *a* quotient map.

Remark 3.6.8. Theorem 3.6.2 holds more generally with $q_X\colon X \to X/{\sim}$ replaced by any quotient map $q\colon X \to Z$ and with $\sim$ replaced by the kernel relation $\sim_q$.

Theorem 3.6.9. Let $f\colon X \to Y$ be a continuous map. If f is a closed map (i.e., takes closed subsets to closed subsets) or an open map (i.e., takes open subsets to open subsets), then f is a quotient map.

Definition 3.6.10. An embedding of a topological space A into a topological space X is an injective continuous map $i\colon A \to X$ such that the induced map $i\colon A \to i(A)$ onto the image of A is a homeomorphism.

For example, the inclusion $i\colon A \to X$ of any subspace $A \subset X$ is an embedding.

Definition 3.6.11. Let $\{X_\alpha\}_{\alpha\in J}$ be a family of spaces and consider the disjoint union $\bigsqcup_{\alpha\in J} X_\alpha$. Let $i_\alpha\colon X_\alpha \to \bigsqcup_{\beta\in J} X_\beta$ be the canonical injections given by $i_\alpha(x) = (\alpha, x)$.

We define the disjoint union topology on $\bigsqcup_{\alpha\in J} X_\alpha$ by declaring that $U \subset \bigsqcup_{\alpha\in J} X_\alpha$ if and only if $i_\alpha^{-1}(U)$ is open in X_α for each $\alpha \in J$; it is the unique topology on $\bigsqcup_{\alpha\in J} X_\alpha$ such that i_α is an embedding for each α. (It is also the finest topology such that each i_α is continuous.)

Definition 3.6.12. Let X, Y, A be topological spaces and fix embeddings $i\colon A \to X$ and $j\colon A \to Y$. Denote by $X \xrightarrow{i_X} X \sqcup Y \xleftarrow{i_Y} Y$ the canonical embeddings into the disjoint union.

The identification space of X and Y along A, denoted $X \cup_A Y$ is the quotient space $(X \sqcup Y)/\sim$, where $\sim$ is the (equivalence relation generated by) the relation $\{(i_X(i(a)), i_Y(j(a))) \mid a \in A\}$.

(Note that both the name and the notation obscure the fact that $X \cup_A Y$ depends on the embeddings i and j.)

Theorem 3.6.13. Let $A, B \subset X$ be closed subspaces of a space X and let $C = A \cap B$ be the intersection, so that we have the inclusions $A \xleftarrow{i} C \xrightarrow{i'} B$.

Then $A \cup_C B \cong X$.

4. Function spaces

4.1. The topology of pointwise convergence.

Definition 4.1.1. Let X and Y be topological spaces. Recall that the set of all functions Y^X is the same as the product space $Y^X = \prod_{x\in X} Y$.

The topology of pointwise convergence on Y^X is by definition just the product topology.

Explicitly, a subbasis for this topology is given by all the sets $B_{x,U} = \{f \in Y^X : f(x) \in U\}$, where $x \in X$ and $U \subset Y$ is open.

Theorem 4.1.2. Let X and Y be topological spaces. Endow the space Y^X with the topology of pointwise convergence.

Then a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X converges to some $f \in Y^X$ if and only if $f_n(x) \xrightarrow{n\to\infty} f(x)$ for each $x \in X$.

4.2. **The supremum metric.**

Definition 4.2.1. Let ∞ be some fixed thing which is not an element of $\mathbb{R}$.

We define the extended real numbers to be the set $R_\infty := \mathbb{R} \cup \{\infty\}$. We extend the standard ordering $\leqslant$ on $\mathbb{R}$ to $\mathbb{R}_\infty$ by declaring $x \leqslant \infty$ for all $x \in \mathbb{R}$.

An extended metric on a set X is a function $d : X \times X \to \mathbb{R}_\infty$ satisfying the usual conditions of a metric: positivity, symmetry, and triangle inequality, but interpreted now with respect to the partial order on $\mathbb{R}_\infty$.

Definition/Theorem 4.2.2. For any extended metric $d : X \times X \to \mathbb{R}_\infty$ on a set X and for any $T > 0$, the function $d^T : X \times X \to \mathbb{R}_\infty$ defined by $d^T(x, y) = \max\{d(x, y), T\}$ is a metric on X.

The metric topology on X associated to T is independent of T and is called the metric topology associated to d. Explicitly, it has as a basis the sets $\mathrm{B}_r(x; d) = \{y \in X \mid d(x, y) < r\}$ for $x \in X$ and $r \in \mathbb{R}$.

Definition/Theorem 4.2.3. Let X be a set and (Y, d) a metric space.

The function $d(f, g) = \sup_{x\in X} d(f(x), g(x))$ defined on Y^X is an extended metric, called the supremum (extended) metric on Y^X .

The topology induced by this extended metric is called the *topology of uniform convergence* (with respect to the metric d).

Showing that this is a metric uses the following lemma:

Lemma 4.2.4. Given bounded non-empty subsets $A, B \subset \mathbb{R}$, we have

$$\sup A + \sup B = \sup\{a + b \mid a \in A, b \in B\}.$$

Remark 4.2.5. With respect to the supremum metric, it is immediate that $f_n \xrightarrow{n\to\infty} f$ if and only if (f_n) converges uniformly to f; i.e.,:

$$\forall \varepsilon > 0 \;\; \exists N > 0 \;\; \forall n \geqslant N \;\; \forall x \in X \; \Big(\big|f(x) - g(x)\big| < \varepsilon \Big).$$

Theorem 4.2.6. Let X be a space and Y a metric space.

Then with respect to the supremum metric, the set $\mathcal{C}(X, Y) \subset Y^X$ of a continuous functions is a closed subspace (i.e., if $f_n \xrightarrow{n\to\infty} f$ uniformly and each f_n is continuous, then f is continuous – this is equivalent since Y^X is metrizable).

4.3. **The compact-open topology.**

Definition 4.3.1. Let X and Y be spaces.

The compact-open topology on the set $\mathcal{C}(X, Y)$ of continuous functions from X to Y is defined by having as a subbasis all sets of the form

$$B_{K,U} = \{f \in \mathcal{C}(X, Y) \mid f(K) \subset U\}$$

where $K \subset X$ is compact and $U \subset Y$ is open.

Theorem 4.3.2. If X is a compact space and (Y, d) is a metric space, then the compact-open topology on $\mathcal{C}(X, Y)$ agrees with the topology of uniform convergence with respect to d; in particular, the topology of uniform convergence on $\mathcal{C}(X, Y)$ does not depend on the metric d.

The proof uses one direction of the following useful lemma:

Lemma 4.3.3. A function $f\colon X \to Y$ between topological spaces is continuous if $f(\bar{A}) \subset \overline{f(A)}$ for all $A \subset X$.

Remark 4.3.4. We now know that if X is compact and Y is a metric space, then a sequence $f_n\colon X \to Y$ converges to some $f\colon X \to Y$ in the compact-open topology if and only if $f_n \xrightarrow{n\to\infty} f$ uniformly.

By similar arguments to the ones above, one can show that for a general X and Y a metric space, we have that convergence $f_n \xrightarrow{n\to\infty} f$ in the compact-open topology is *uniform convergence on compact sets*, i.e., for each compact $K \subset X$, we have that $f_n|_K \xrightarrow{n\to\infty} f|_K$ uniformly.

4.4. **Continuous functions into functions spaces.**

Theorem 4.4.1. Let X, Y, Z be spaces and let $f\colon X \times Y \to Z$ be a continuous function.

Then for each $x \in X$, the function $F(x)\colon Y \to Z$ defined by $F(x)(y) = f(x, y)$ is continuous, and the resulting function $F\colon X \to \mathcal{C}(Y, Z)$ is continuous as well (where we equip $\mathcal{C}(Y, Z)$ with the compact-open topology).

Definition 4.4.2. A space X is locally compact Hausdorff if it is Hausdorff and, for each neighbourhoods U of a point $x \in X$, there is a neighbourhood $V \subset X$ such that $\bar{V}$ is compact and $\bar{V} \subset X$.

Remark 4.4.3. As the name suggests, there is a weaker notion of local compactness such that X is locally compact Hausdorff if and only if it is locally compact and Hausdorff.

Theorem 4.4.4. Any closed subspace of $\mathbb{R}^n$ is locally compact Hausdorff (for any n).

Theorem 4.4.5. Let X, Y be spaces and assume X is locally compact Hausdorff. Then the map $\mathrm{ev}\colon \mathcal{C}(X, Y) \times X \to Y$ given by $\mathrm{ev}(f, x) = f(x)$ is continuous (where we equip $\mathcal{C}(Y, Z)$ with the compact-open topology).

Corollary 4.4.6. Let X, Y, Z be spaces and assume X is locally compact Hausdorff.

Then given a continuous map $F\colon X \to \mathcal{C}(Y, Z)$ (where we equip $\mathcal{C}(Y, Z)$ with the compact-open topology), the map $f\colon X \times Y \to Z$ defined by $f(x, y) = F(x)(y)$ is continuous.

5. A bit of algebraic topology

5.1. **Homotopy.**

Definition 5.1.1. Given spaces X and Y and continuous maps $f, g\colon X \to Y$, a homotopy from X to Y is a map $H\colon X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

If there exists a homotopy from f to g, we say that f and g are homotopic and write $f \sim g$.

Remark 5.1.2. Any homotopy $H\colon X \times [0, 1] \to Y$ from f to g defines a path $[0, 1] \to \mathcal{C}(X, Y)$ from f to g. It is common to write $h_t\colon X \to Y$ for $h(t)$.

If X is locally compact Hausdorff, then the converse also hoods, and hence a homotopy from f to g can simply be defined as a path from f to g in $\mathcal{C}(X, Y)$.

Theorem 5.1.3. Given spaces X and Y, the relation on $\mathcal{C}(X, Y)$ of being homotopic is an equivalence relation.

Definition 5.1.4. Let α and β be paths in a space X from $x_0 \in X$ to $x_1 \in X$; i.e., $\alpha(0) = \beta(0) = x_0$ and $\alpha(0) = \beta_1) = x_1$. A homotopy relative to the endpoints (or just homotopy rel endpoints, or even just homotopy) between α and β is a continuous map $H\colon [0,1] \times [0,1] \to X$ such that

$$\begin{cases} H(s,0) = \alpha(s), \ \forall s \in [0,1] \\ H(s,1) = \beta(s), \ \forall s \in [0,1] \\ H(0,t) = x_0, \ \forall t \in [0,1] \\ H(1,t) = x_1, \ \forall t \in [0,1] \end{cases}$$

Equivalently (since $[0,1]$ is locally compact Hausdorff), a homotopy rel endpoints from α to β is a path from α to β in the space of paths in X from x_0 to x_1, denoted $\mathrm{P}_{x_0,X_0}(X)$ and defined as

$$\mathrm{P}_{x_0,x_1}(X) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = x_0 \text{ and } \gamma(1) = x_1\} \subset \mathcal{C}([0,1], X)$$

If such a homotopy exists, we say that α_0 and α_1 are homotopic rel endpoints (or just homotopic), and we write $\alpha_0 \sim \alpha_1$.

Theorem 5.1.5. The relation on $P_{x_0,x_1}(X)$ of being homotopic rel endpoints is an equivalence relation.

Definition 5.1.6. A space X is simply connected if for any $x, y \in X$, there is a unique homotopy class of paths from x to y.

Definition 5.1.7. A set $U \subset \mathbb{R}^n$ is convex if $\mathbf{x} + t(\mathbf{x} - \mathbf{y}) \in U$ for each $\mathbf{x}, \mathbf{y} \in U$ and each $t \in [0,1]$.

Theorem 5.1.8. $\mathbb{R}^n$ is simply connected.

More generally, any convex subset $U \subset \mathbb{R}^n$ is simply connected.

5.2. **The fundamental group.**

Definition 5.2.1. Let X be a space, and let $\alpha \in \mathrm{P}_{x_0,x_1}(X)$ and $\beta \in \mathrm{P}_{x_1,x_2}(X)$.

The composition of α and β is the path $\alpha * \beta\colon \in \mathrm{P}_{x_0,x_2}$ defined by

$$(\alpha \cdot \beta)(s) = \begin{cases} \alpha(2s) \text{ if } 0 \leqslant s \leqslant \frac{1}{2} \\ \beta(2s-1) \text{ if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

(That $\alpha * \beta$ is continuous follows from the pasting lemma.)

Definition 5.2.2. Fix a point x_0 in a space X.

The fundamental group of X based at x_0 is the set of equivalence classes

$$\pi_1(X, x_0) = \{\alpha\colon [0,1] \to X \mid \alpha(0) = \alpha(1) = x_0\}/\sim.$$

In other words, $\pi_1(X, x_0)$ is the set of path components of the loop space of X at x_0 $\Omega_{x_0}(X) := \mathrm{P}_{x_0,x_0}(X)$.

Definition 5.2.3. A group is a pair $(G, \cdot)$ consisting of a set G and an operation $\cdot\colon G \times G \to G$ satisfying:

(i) (*associativity*) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.
(ii) (*identity element*) There is a unique element $e \in G$ such that $e \cdot g = g \cdot e = g$ for all $g \in G$.
(iii) (*inverses*) For each $g \in G$, there is a unique element g^{-1} such that $g \cdot g^{-1} = g^{-1} \cdot g = e$ for all $g \in G$.

As usual, we often conflate a group $(G, \cdot)$ with its underlying set G.

Theorem 5.2.4. There is a unique group structure $\cdot$ on $\pi_1(X, x_0)$ defined by the property that $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ for all $\alpha, \beta \in \Omega_{x_0}(X)$.

Here are the key steps of the proof of the Theorem. First we have to show that the law is well defined.

Lemma 5.2.5. If $\alpha \sim \alpha'$, $\beta \sim \beta'$ and $\alpha(1) = \beta(0)$, then $\alpha \cdot \beta \sim \alpha' \cdot \beta'$.

The remaining verifications all rely on the following:

Lemma 5.2.6. Given any paths γ and δ in a space Y (with the same endpoints) and a continuous map $f\colon Y \to Z$, if $\gamma \sim \delta$, then $f \circ \gamma \sim f \circ \delta$.

We next have to show that the law is associative.

Lemma 5.2.7. If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then $(\alpha \cdot \beta) \cdot \gamma \sim \alpha \cdot (\beta \cdot \gamma)$.

We then have to find an identity element. For this, given $x \in X$, let $\mathrm{const}_x\colon [0,1] \to X$ denote the *constant path* defined by $\mathrm{const}_x(s) = x$ for every $s \in [0,1]$.

Lemma 5.2.8. $\mathrm{const}_{\alpha(0)} \cdot \alpha \sim \alpha \sim \alpha \cdot \mathrm{const}_{\alpha(1)}$ for every path α.

Finally, we need an inverse. The *opposite* of a path α, denoted α^{-1}, is the path obtained by traveling backwards, namely defined by $\alpha^{-1}(s) = \alpha(1-s)$.

Lemma 5.2.9. $\alpha \cdot \alpha^{-1} \sim \mathrm{const}_{\alpha(0)}$ and $\alpha^{-1} \cdot \alpha \sim \mathrm{const}_{\alpha(1)}$ for every path α.

Definition 5.2.10. Given groups G and H, a homomorphism from G to H is a map $\varphi\colon G \to H$ such that $\varphi(e_G) = e_H$ (where e_G and e_H are the identity elements of G and H) and $\varphi(g \cdot h) = \varphi(g) \cdot \varphi(h)$ for all $g, h \in G$.

If φ is further a bijection, we say that φ is an isomorphism and that G and H are isomorphic, and write $G \cong H$. (Note in this case that φ^{-1} is automatically an isomorphism as well.)

Theorem 5.2.11. If X is a space and $x_0, x_1 \in X$ are in the same path-component, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

Theorem 5.2.12. If X is path-connected and $\pi_1(X, x_0) = \{e\}$ for some (and hence any) $x_0 \in X$, then X is simply connected.

5.3. **Homomorphisms induced by maps.**

Definition/Theorem 5.3.1. Given a continuous map $f\colon X \to Y$, the map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $f_*([\alpha]) = [f \circ \alpha]$, where $y_0 = f(x_0)$ is well-defined and is a homomorphism.

It is called the group homomorphism induced by f.

The following is called the *functoriality* property of the induced homomorphism.

Theorem 5.3.2. Given continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, with $y_0 = f(x_0)$ and $z_0 = g(y_0)$, we have

$$(g \circ f)_* = g_* \circ f_*\colon \pi_1(X, x_0) \to \pi_1(Z, z_0)$$

Also, we have

$$(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x_0)}\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$$

Functoriality implies the fundamental property of *isomorphism-invariance*:

Corollary 5.3.3. If X and Y are homeomorphic and both path-connected, then for any $x_0 \in X$ and $y_0 \in Y$, we have $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

Equivalently, if X and Y (are path-connected and) have non-isomorphic fundamental groups, then they are not homeomorphic!

We can strengthen the isomorphism-invariance and show invariance under *homotopy equivalence*.

Theorem 5.3.4. Let X and Y be spaces, and let $x_0 \in X$. Let $f, g\colon X \to Y$ be such that $f(x_0) = g(x_0) =: y_0$.

If there is a homotopy H from f to g such that $H(x_0, t) = y_0$ for all t, then the two homomorphisms f_* and $g_*\colon \pi(X, x_0) \to \pi_1(Y, y_0)$ are equal.

Definition 5.3.5. A retraction of a topological space X onto a subspace $Y \subset X$ is a continuous map $f\colon X \to Y$ such that $f|_Y = \mathrm{id}_Y$. Y is then said to be a retract of X.

The retraction $f\colon X \to Y$ is a deformation retraction if there exists a homotopy h_t from id_X to $f \circ i_Y$ (where $i_Y\colon Y \hookrightarrow X$ is the inclusion) which fixes Y, i.e. $h_t|_Y = \mathrm{id}_Y$ for all Y. (Note that this is sometimes called a *strong deformation retraction.*) In this case, Y is said to be a deformation retract of X.

Theorem 5.3.6. If $Y \subset X$ is a deformation retract, then $\pi_1(X, x_0) \cong \pi_1(Y, x_0)$ for any $x_0 \in Y$.

Corollary 5.3.7. For any $n \geqslant 1$ and any $x_0 \in \mathrm{S}^n$, we have $\pi_1(\mathrm{S}^n, x_0) \cong \pi_1(\mathbb{R}^{n+1} \backslash \{0\}, X_0)$.

5.4. Fundamental group of the circle.

Theorem 5.4.1. $\pi_1(\mathrm{S}^1, x_0) \cong \mathbb{Z}$ for any $x_0 \in \mathrm{S}^1$.

5.5. Baby Seifert-Van Kampen theorem.

Theorem 5.5.1. Let X be a space and let $U, V \subset X$ be an open covering with non-empty intersection, and let $x_0 \in U \cap V$.

If $\pi_1(U, x_0) = \{e\}$ and $\pi_1(V, x_0) = \{e\}$, then $\pi_1(X, x_0) = \{e\}$.

Corollary 5.5.2. $\pi_1(\mathrm{S}^n, x_0)$ is trivial for $n \geqslant 2$.

Theorem 5.5.3. $\mathbb{R}^2 \ncong \mathbb{R}^n$ for $n > 2$.

5.6. Products.

Theorem 5.6.1. Let the product $X \times Y$ of the two topological spaces X and Y be endowed with the product topology, and let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the two projection maps defined by $p(x, y) = x$ and $q(x, y) = y$. Then the product map

$$\langle p_*, q_* \rangle\colon \pi_1\big(X \times Y, (x_0, y_0)\big) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

given by $\langle p_*, q_* \rangle(g) = (p_*(g), q_*(g))$ is a group isomorphism.

Corollary 5.6.2. Defining the n-torus $\mathbb{T}^n$ to be the n-fold product $(\mathrm{S}^1)^n$, we have $\pi_1(\mathbb{T}^n, x_0) \cong \mathbb{Z}^n$ (for any $x_0 \in \mathbb{T}^n$).

Corollary 5.6.3. $\mathbb{T}^2 \ncong \mathrm{S}^2$.