Project assignment

Math 440, Fall 2022

1. Overview

The goal of your project assignment is to explore an aspect of topology beyond the topics covered in class. It will be necessary to use the tools and properties of topological spaces that we have developed, such as: compactness, connectedness, the notion of a continuous function and homeomorphisms, constructions of topological spaces, etc. You will work with groups of 2-3 students. As part of this assignment, you will write a short paper (4-5 pages), and you will give a 20 minute presentation.

2. Grading

The final project will be worth 75 points total, split between a paper (4-6 pages maximum) and presentation (20 minute group presentation) component:

(1) **10 points:** *Outline submission (due Nov. 11).* Requirements: a paragraph stating your plan, followed by 1 or 2 page outline detailing your paper plan: sections (ideally 3-4), mention of examples you plan to study, results you will state, results you will prove or sketch, and motivation you will discuss; in the order it appears. Your grade will be based only on completeness/reasonable effort.

(2) **10 points:** *First draft submission (due Nov. 21).* You do not need to exactly follow your outline, in case you decided to make changes. Your grade for this part will be based on completion; did you make a reasonable effort to fill out all of the sections you proposed in your outline (suitably adjusted in case you made changes)? It is okay if some specific lemmas, etc. are not fleshed out, as long as the paper is structurally fleshed out.

(3) **30 points:** *Final draft submission (4-6 pages maximum, due Dec. 2).* Here you will be graded on the quality of your submission, according to roughly 4 categories:

   (a) *Content:* Is a reasonable breadth of content covered, with examples?

   (b) *Mathematical writing:* Are proofs written in understandable English; were common "errors in mathematical writing" avoided? (see e.g., `http://www.math.uconn.edu/~kconrad/math216/mathwriting.pdf`)

   (c) *Exposition:* Is there a sensible overview at the beginning of what is going to happen in the document? Are there transitions between results? Explanations for why intermediate technical results are used? Motivation for why one is doing various things? Did your group judiciously choose what to leave in and what to leave out? Are examples given to explain difficult concepts?

   (d) *Organization:* Are different parts of content broken up into sensible sections/subsections as needed? Are proofs of results split into reasonable Lemmas, Propositions, etc.?

(4) **25 points:** *20 min group presentation.* Here you will be graded on organization (both of your talk and board work), exposition (as above), and content.

Note: all members of the group should be part of the presentation, at least for a couple minutes each.

Note: All members of a group will receive the same score.

3. Timeline/Due Dates

(1) **Selecting your group and project**: The deadline for selecting your group and projects is **Friday, November 4**. At least one person from each group should e-mail me with your group and your project choice (or tell me in person).
(2) **Outline:** A proposed project outline is due on **Friday, November 11**. The goal of this outline is for you to flesh out your project plan and proposed references, in bullet point or outline form, and clarify any confusions you might have about what is okay to cover and what is okay to skip. (After this submission, you will receive feedback from me as to whether the outline seems like a reasonable plan for your paper or whether you may wish to consider adding/removing sections).
(3) **First draft:** The first draft is due **Monday, November 21**. (After this submission, you will receive feedback from on your draft).
(4) **Final draft:** The final draft is due **Friday, December 2**.
(5) **Presentation:** You will give a 20 minute lecture/presentation on your topic, jointly with your group (all members must say something). Final presentations will occur on the last 2 days of class: **November 30 and December 2**.

All submissions should be made by email.

4. Typesetting

It is required that you type your paper using software of your choice. You are strongly encouraged (but it is not mandatory) to use LaTeX to write your paper.

5. On expository writing

The aim of the written portion of the assignment is for you to practice writing mathematical exposition; it is important that your writing be precise yet understandable to someone else in your class with a comparable background. Understandability means several things:

(1) From an orthographic standpoint, you should avoid the use of any mathematical shorthand symbols like $\forall$ and $\exists$, unless these are appearing in a complete formula; instead write "for all" and "there exists." Always write in complete sentences, and explain conclusions to arguments.
(2) From a stylistic standpoint, you may choose to first give conceptual explanations of a fact before giving a rigorous proof. Or going even further, at times you may choose to omit a certain (inessential) proof, but only give a conceptual explanation of it.

6. On presentations

Think carefully about how you want to structure your class time. What are the most important ideas you are trying to convey; what is less important? What results are worth stating but completely omitting the proof of, so that you can get to other key arguments? In case you run out of time, what is okay to skip? How will you split time among the entire group? I would suggest you give one or two practice talks amongst your groups (or to another group); in particular, develop familiarity with using the whiteboard. How does your board look from the back of class?

7. Project ideas

You are welcome to choose one of the following list of projects, or come up with another project with instructor approval. Broadly, the idea is to explore an area with a collection of definitions, examples (with proof), and main results (with at least partial proofs/proof explanations – complete proofs if they are not too long).

Below are some project ideas, and broad suggestions for what details to include. Of course, you are encouraged to write to me or ask me in office hours if you need help familiarizing yourself with your topic.

Note: You are not required to exactly follow the suggestions; feel free to deviate as desired.

(1) **Metrization theorems:** When is a topological space $(X, \mathcal{T}_X)$ metrizable? We have seen necessary conditions: a X had better be Hausdorff to have any hope of being metrizable. But this conditions are not sufficient: it is not true that if X is Hausdorff, then X must be metrizable. (Find an example!).

The goal of this project is to study the metrizability question: specifically, give sufficient, or better, necessary and sufficient conditions for metrizability of a topological space. The relevant possible results are called Urysohn's metrization theorem (which is sufficient) and/or the Nagata-Smirnov theorem (which is necessary and sufficient) and/or the Smirnov metrization theorem (which is necessary and sufficient, and follows from Nagata-Smirnov); these are both described in Munkres (in Ch 4, §34 and Ch 6, §§40,42).

You do not need to describe all of these, and if you describe more than one, you do not need to give complete details for all of them (but should focus on giving more details for at least one of the theorems, and whatever definitions are necessary to state the other theorems).

Complete proofs are not necessary, but some summary of the proof and relevant definitions are. Examples are also helpful; you may for instance pick one or two of the exercises.

(2) **Manifold theory:** A (topological) n-manifold is, roughly, a topological space which is locally homeomorphic to an open subset of R^n.

Manifolds are fundamental objects in geometry and topology, and appear frequently in other disciplines as well.

There are at least three possible projects here:

(a) **An introduction to manifolds and their classification.** Give the definition of an n-manifold, prove basic properties of manifolds, and produce examples (with proof) of some compact connected 1 and 2-dimensional manifolds (in dimension 1, the circle, in dimension 2, the sphere, surfaces of genus g, and real projective space).
State (without proof) the classification of 1 and 2-dimensional manifolds up to homeomorphism, and possibly give a few words (paragraph at most) of intuition about why these classifications are true.

(b) **Manifolds and their embeddings.** Give the definition and some basic examples of an n-manifold, and prove basic properties of manifolds. Define an embedding of manifolds $M \to N$, and give some basic examples. Sketch a proof that any compact manifold M (of some dimension m) embeds into R^N for some $N > 0$; this is *Whitney's embedding theorem*, discussed in Ch 4, §36 of Munkres.

(c) **An introduction to knot theory.** A knot is an embedding of the compact 1-manifold s^1 into 3-dimensional Euclidean space R^3. Roughly, speaking, the image of a knot is any closed loop you can draw in R^3 which does not intersect itself.

Any two knots are homeomorphic as topological spaces (equipped with the subspace topology from R^3), as they are each homeomorphic to S^1. However, there is another notion of being "the same" that is relevant for knots (or more generally, any embeddings): that of isotopy.

Roughly speaking, two knots K_0, K_1 are isotopic if one can be continuously deformed into the other without creating any self-crossings.[1]

A remarkable fact is that there are many pairs of knots that are not isotopic, and to this date an exact classification of isotopy classes of knots is unknown! (Knot theory is an active area of research in mathematics that studies such questions).

The goal of this project is to initiate the study of classifying isotopy classes of knots: define knots (you'll need to define what an "embedding" $\mathrm{S}^1 \to \mathbb{R}^3$ is, but not more general types of embeddings) and isotopies of knots, give a brief survey of the classification question, and find with proof two non-isotopic knots (though you may assume whatever fundamental facts about knot theory you wish)

(3) **Topological dimension of a space:** Using open-covers, define a number called the topological dimension of any topological space M. The crowing result of dimension theory gives an answer to a very basic question about topological spaces (and one closely related to the metrizability theorems, and also to Whitney's embedding theorem): Which topological spaces are homeomorphic to subspaces of $\mathbb{R}^n$?

The goal of this project is to explore the definition of dimension (given in Munkres Ch 8, §50), prove some of its basic properties, give examples (solve some of the exercises in Ch 8, §50), and say something about the proof of the aforementioned theorem.

(4) **Methods of constructing topological spaces.** Explore the notion of CW complexes (a good reference is Hatcher's *Algebraic Topology*, Chapter 0), and/or simplicial complexes (and say what a triangulation of a topological space is).

Give many examples (the sphere, torus, higher genus surfaces, for instance); discuss properties of these spaces (e.g., when are they compact or connected?).

(5) **Space-filling curves:** We have proven in class that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic; that is, there is no continuous function $\mathbb{R} \to \mathbb{R}^2$ with continuous inverse.

Similarly, the line segment $[0,1]$ and the square $[0,1]^2$ are not homeomorphic; since they are compact and Hausdorff, this also means that there is no continuous bijection $f\colon [0,1] \to [0,1]^2$.[2] On the other hand, when considered as sets, $[0,1]$ and $[0,1]^2$ (resp. $\mathbb{R}$ and $\mathbb{R}^2$) are in bijection (why? construct one bijection as part of this project).

One can ask an intermediate question: is there is a continuous surjection $[0,1] \to [0,1]^2$?

Bizarrely, the answer is yes! Such maps are called space-filling curves; the first such curve was discovered by Peano in 1890. The goal of this project is to give an overview of the above discussion and then construct, with proof, one such space filling curve $f\colon [0,1] \to [0,1]^2$, for instance the one detailed in Munkres Ch 7 §44.

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[1] Formally, two knots $f, g : \mathrm{S}^1 \to \mathbb{R}^3$ are isotopic if one can find a continuous family of embeddings $\{f_t\}_{t\in[0,1]}$ which equals f at t = 0 and g at t = 1; "continuous family of embeddings" means that (i) f_t should be an embedding for all $t \in [0,1]$, and (ii) (continuity) the map $\mathrm{S}^1 \times [0,1] \to \mathbb{R}^3$ sending (s,t) to $f_t(s)$ is continuous.

[2] The proof is the same as the case of $\mathbb{R}$ and $\mathbb{R}^2$; roughly note that if we remove the point $1/2$, then $[0,1] - \{1/2\}$ is no longer connected; on the other hand, $[0,1]^2$ minus any point is always connected.

You will probably need to prove some theorems about completeness of metric spaces; the desired f is constructed by a limiting procedure; one constructs approximations f_n for each n and argues that a limit must exist (by completeness).

(6) **Invariants of topological spaces, and their applications:** An invariant of a topological space X is some sort of object $k(X)$ one assigns to X, which is (a) hopefully easy to compute (for instance, a number), and is (b) unchanged by homeomorphism (meaning that if $X \cong Y$, then $k(X) = k(Y)$).

Invariants give a tool for understanding whether spaces are homeomorphic; more precisely, they give us tools for understanding when spaces are not homeomorphic (since if $k(X) \neq k(Y)$, then X cannot be homeomorphic to Y).

Famous invariants include numerical invariants such as the *Euler characteristic* $\chi(X)$ and the *Betti numbers* $\beta_i(X)$, and more sophisticated "algebraic" invariants, such as the *homology groups* $\mathrm{H}_i(X)$ and the *fundamental group* $\pi_1(X)$ (which we may discuss in class at the end of the semester).

These last two invariants are groups[3]and are invariant in the sense that, for example, if $X \cong Y$, then $\pi_1(X) \cong \pi_1(Y)$ (isomorphism of groups).

The goal of this project, loosely defined, is to give the definition of one (or more) of these invariants, and study applications thereof. Here are some suggested applications, which can be found in the textbook; each could consist of a different project:

(a) **Homotopy equivalences.** Study the notion of a deformation retract, and more generally a homotopy equivalence, which are weaker notions than two spaces being homeomorphic (give examples). Prove that π_1 is unchanged under homotopy equivalences, and use this to compute π_1 in some examples. Good references include Hatcher's Algebraic Topology book (Chapter 0, available online), or Munkres Ch 9, §58.

(b) **Fundamental theorem of algebra.** The fundamental theorem of algebra states that any polynomial $a_n z^n + \cdots + a_1 z + a_0$, with $a_i in \mathbb{C}$ complex numbers, has a zero (in the complex numbers). It is a remarkable fact that this theorem can be proven using topology, specifically the fundamental group π_1. Give an account of this proof, using a reference of your choice (one account is in Munkres Ch 9, §56).

[3]Recall that a group is a set G with a multiplication operation $\cdot\colon G \times G \to G$ satisfying some familiar properties (consult references).