Y IS A LEAST FIXED POINT COMBINATOR

JOSEPH HELFER

Abstract. The theory of recursive functions is related in a well-known way to the notion of *least fixed points*, by endowing a set of partial functions with an ordering in terms of their domain of definition. When terms in the pure λ-calculus are considered as partial functions on the set of reduced λ-terms, they inherit such a partial order. We prove that Curry's well-known fixed point combinator Y produces least fixed points with respect to this partial order.

1. Introduction

Curry's "paradoxical combinator" Y in the pure lambda calculus, defined as

$$\mathrm{Y} = \lambda f.\big(\lambda g.f(gg)\big)\big(\lambda g.f(gg)\big), \tag{1}$$

is such that $\mathrm{Y}F$ is a fixed point of F for any lambda term F, in the sense that $F(\mathrm{Y}F)$ and $\mathrm{Y}F$ are β-equivalent. Y is sometimes called the "least fixed point operator" because there are models of the pure lambda calculus in which the domain is equipped with a partial order, and in which it the interpretation of Y is a function taking each element of the domain to a least fixed point with respect to that partial order (see, e.g., [Bar84, Theorem 19.3.4]).

However, the set of lambda terms itself can be equipped with the usual partial order (or rather, preorder) on partial functions by declaring that $F \leq G$ for lambda terms F and G if and only if, for all lambda terms M, if FM has a β-normal form, then GX has the same β-normal form (Definition 2). One can then ask whether $\mathrm{Y}F$ is always minimal with respect to this order among fixed points of F. Surprisingly, this questions does not seem to have appeared in the literature. In this note, we answer it in the affirmative.

In §4, we briefly discuss the question of other fixed-point combinators.

Acknowledgements: We thank Henk Barendregt for helpful discussions concerning this paper.

2. Preliminaries

Let V be an infinite set (of "variables"). We write Λ for the set of **lambda terms** with variables in V, up to α-equivalence. We will refer to the elements of Λ simply as "lambda terms" (or just "terms") rather than "lambda terms up to α-equivalence".

Rather than recalling the explicit definition of Λ (for which see, e.g., [Bar84, Chapter 2]), we state certain properties which uniquely characterize it. More precisely, we will state certain properties of (a) Λ, together with its operations $V \to \Lambda$ (atomic terms), $\Lambda \times \Lambda \to \Lambda$ (application), and $V \times \Lambda \to \Lambda$ (abstraction), (b) the set $\mathrm{FV}(M) \subset V$ of **free variables** of a lambda term M, and (c) the operation of **substitution** $M[v := N]$ of lambda terms (M and N terms and v a variable). These properties determine Λ together with the mentioned operations on it uniquely up to isomorphism, and determine the sets $\mathrm{FV}(t)$ and the substitution operation uniquely:

Date: March 31, 2025.

(a) Every term is of exactly one of the forms (i) v with $v \in V$, (ii) MN for $M, N \in \Lambda$, or (iii) $\lambda v.M$ with $v \in V$ and $M \in \Lambda$. In case (i), v is uniquely determined, and in case (ii), M and N are uniquely determined. For case (iii), if $\lambda u.M = \lambda v.N$, then $N = M[u := v]$. Moreover Λ is the least subset of Λ closed under the operations (i)-(iii), i.e., we can use structural induction and recursion on Λ.

(b) The set $\mathrm{FV}(M) \subset V$ of free variables of a term M is determined by (i) $\mathrm{FV}(v) = \{v\}$ for $v \in V$, (ii) $\mathrm{FV}(MN) = \mathrm{FV}(M) \cup \mathrm{FV}(N)$, and (iii) $\mathrm{FV}(\lambda u.M) = \mathrm{FV}(M) - \{u\}$.

(c) The substitution operation $M[v := N]$ is determined by (i) $v[v := M] = M$ and $u[v := M] = u$ for $u \in V - \{v\}$, (ii) $(M_1 M_2)[v := N] = (M_1[v := N])(M_2[v := N])$, and (iii) $(\lambda u.M_1)[v := s] = \lambda u.(M_1[v := s])$ for $u \in V - (\mathrm{FV}(s) \cup \{v\})$.

Note that if $u \in \mathrm{FV}(s) \cup \{v\}$, we can always take some $u' \in V - (\mathrm{FV}(s) \cup \{v\})$, and set $M_1' = M_1[u := u']$, and we then have $\lambda u.M_1 = \lambda u'.M_1'$, so the clauses (i)-(iii) indeed suffice to determine the substitution operation.

For a relation $R \subset \Lambda \times \Lambda$, we write $M \sim_R N$ for $(M, N) \in R$. We say that R is **compatible** (see [Bar84, p. 3.1.1]) if given terms M, M', N, N' with $M \sim_R M'$ and $N \sim_R N'$, then $MN \sim_R M'N$, $MN \sim_R MN'$, and $\lambda v.N \sim_R \lambda v.N'$ for $v \in V$. For a relation R, we write $\to_R$ (**one-step *R*-reduction**)) for the least compatible relation containing R, we write $\twoheadrightarrow_R$ (***R*-reduction**) for the least preorder containing $\to_R$, and $\approx_R$ (***R*-equivalence**) for the least equivalence relation containing $\twoheadrightarrow_R$ (or equivalently, containing $\to_R$). It follows that $\twoheadrightarrow_R$ and $\approx_R$ are themselves compatible, and are hence the least compatible preorder and equivalence relation containing R, respectively.

The relations $\beta, \eta \subset \Lambda \times \Lambda$ are defined as

$$\beta = \{\big((\lambda v.M)N, M[v := N]\big) \mid v \in V;\ M, N \in \Lambda\} \quad \text{and} \quad \eta = \{\big((\lambda v.Mv), M\big) \mid v \in V;\ M \in \Lambda\},$$

and we define $\beta\eta = \beta \cup \eta$. We thus obtain the relations $\to_\beta$, $\twoheadrightarrow_\beta$, and $\approx_\beta$ of one-step β-reduction, β-reduction, and β-equivalence, and likewise $\to_{\beta\eta}$, $\twoheadrightarrow_{\beta\eta}$, and $\approx_{\beta\eta}$.

Lemma 1. *If R is either of β or $\beta\eta$ (or more generally if $\beta \subset R$) then*

$$M \approx_R M' \Rightarrow M[v := N] \approx_R M'[v := N]$$

for all $M, M', N \in \Lambda$ and $v \in V$.

Proof. Assuming $M \approx_R M'$, we have $M[v := N] \approx_R (\lambda v.M)N \approx_R (\lambda v.M')N \approx_R M'[v := N]$. □

For the remainder of this section, fix a relation $R \subset \Lambda \times \Lambda$.

A term M is in ***R*-reduced**, if there is no term N with $M \to_R N$. An ***R*-normal form** of a term M is an R-reduced term N with $M \approx_R N$. We say that R has the **Church-Rosser property** if for all $M, N, N' \in \Lambda$, if $M \twoheadrightarrow_R N$ and $M \twoheadrightarrow_R N'$, then there is $L \in \Lambda$ with $N \twoheadrightarrow_R L$ and $N' \twoheadrightarrow_R L$. This implies that every term has at most one R-normal form, and that if N is the R-normal form of M, then $M \twoheadrightarrow_R N$. By the Church-Rosser theorem, β and $\beta\eta$ both have the Church-Rosser property [Bar84, §11.1].

The **Y-combinator** is the lambda term Y defined in (1).

Definition 2. We define a preorder $\leq_R$ on terms by putting, for $F, G \in \Lambda$:

$$F \leq_R G \iff (\forall M \in \Lambda.\ FM \text{ has an } R\text{-normal form} \Rightarrow GM \text{ has the same } R\text{-normal form}).$$

3. The proof

Theorem. *(Y yields R-minimal fixed points.) Let R be either β or $\beta\eta$. Then*

$$\forall F, M \in \Lambda.\ FM \approx_R M \Rightarrow \mathrm{Y}F \leq_R M.$$

We first give an outline of the proof. Given a fixed point M of F, and a term N such that $(\mathrm{Y}F)N$ has a normal form N', we must show that MN has the same normal form. The idea is that the reduction $(\mathrm{Y}F)N \twoheadrightarrow_R N'$ should use "nothing about" $\mathrm{Y}F$ *except* that it is a fixed point F. Thus, we introduce a new variable y which we consider as a "formal fixed point of y" by introducing a new reduction rule $y \to Fy$, and show (i) that the reduction $FN \twoheadrightarrow_R N'$ gives rise to a parallel reduction $yN \twoheadrightarrow N'$, and (ii) that the reduction $yN \twoheadrightarrow N'$ gives rise to a parallel reduction $MN \to N'$. This second step is easy, by substituting M for y in the reduction $yN \twoheadrightarrow N'$.

Step (i) is a little trickier, since in the reduction $(\mathrm{Y}F)N \twoheadrightarrow N'$, we can make reductions *inside of* $\mathrm{Y}F$, whereas we cannot make reductions "inside of y". This is taken care of by allowing each instance of y appearing in the reduction $yN \twoheadrightarrow N'$ to "correspond" to an arbitrary *reduction* of $\mathrm{Y}F$, and keeping track of which instances of y correspond to which such reductions.

We now proceed with the proof. Henceforth, let R be one of β or $\beta\eta$.

Definition 3. For any $y \in V$ and $F \in \Lambda$, we define the relation $S_{F,y} \subset \Lambda \times \Lambda$ by $S_{F,y} = \{(y, Fy)\}$, and we set $R_{F,y} = R \cup S_{F,y}$. We thus have the relations $\to_{S_{F,y}}$, $\twoheadrightarrow_{S_{F,y}}$, and $\approx_{S_{F,y}}$ of one-step $S_{f,z}$-reduction, $S_{F,z}$-reduction, and $S_{F,z}$-equivalence.

Lemma 4. *Let $F \in \Lambda$ and $y \in V - \mathrm{FV}(F)$. Then*

$$FM \approx_R M \wedge N \approx_{R_{F,y}} N' \Rightarrow N[y := M] \approx_R N'[y := M]$$

for any $M, N, N' \in \Lambda$.

Proof. Since $\sim_{R_{F,y}}$ is the least compatible equivalence relation containing $R_{F,y}$, and since $\{(N, N') \mid N[y := M] \approx_R N'[y := M]\}$ is itself a compatible equivalence relation, it suffices to prove the conclusion in the case where $N \sim_{R_{F,y}} N'$.

If $N \sim_R N'$ (or more generally if $N \approx_R N'$), then the conclusion follows immediately from Lemma 1. If $N \sim_{S_{F,y}} N'$, then $N = y$ and $N' = Fy$, hence $N[y := M] = M \approx_R FM = N'[y := M]$ since $y \notin \mathrm{FV}(F)$. □

Lemma 5. *Let $F \in \Lambda$ and $z \in V$. If $M \in \Lambda$ is $R_{F,y}$-reduced, then $y \notin \mathrm{FV}(M)$.*

Proof. By induction on M, using that y is not $R_{F,y}$-reduced. □

We now dispense with "step (ii)" from the proof outline above.

Proposition 6. *Let $F, M, N \in \Lambda$ and suppose $FM \approx_R M$. For any $y \in V - \mathrm{FV}(F) \cup \mathrm{FV}(M) \cup \mathrm{FV}(N)$, if yN has an $R_{F,y}$-normal form, then MN has the same R-normal form.*

Proof. If yN has an $R_{F,y}$-normal form, then there is a sequence $yN = N_0, \ldots, N_n$ where N_n is $R_{F,y}$-reduced and $N_i \to_{R_{F,y}} N_{i+1}$ for each $i < n$. Now consider the sequence $N_0[y := M], \ldots, N_n[y := M]$. We have $N_0[y := M] = MN$, and by Lemma 5, $N_n[y := s] = N_n$, which is $R_{F,y}$-reduced and hence R-reduced. By Lemma 4, $N_i[y := M] \approx_R N_{i+1}[y := M]$ for all i, and hence $MN \approx_R N_n$, as desired. □

We now proceed to "step (i)" in the above proof outline. We begin with some preliminary definitions. As indicated in the outline, we will need to get a handle on what arbitrary R-reductions of $\mathrm{Y}F$ look like.

Definition 7. Given $M, N \in \Lambda$, let $\mathrm{Y}_{M,N}$ be the term $\big(\lambda g.M(gg)\big)\big(\lambda g.N(gg)\big)$, where $g \notin \mathrm{FV}(M) \cup \mathrm{FV}(N)$, so that $\mathrm{Y} = \lambda f.\mathrm{Y}_{f,f}$.

For $n \geq 0$ and $M, N \in \Lambda$, we define $M^{(n)}N$ recursively by $M^{(0)}N = N$ and $M^{(n+1)}N = M(M^{(n)}N)$. For $n \geq 0$, define Y_n to be the term $\lambda f.f^{(n)}\mathrm{Y}_{f,f}$, so that $\mathrm{Y}_0 = \mathrm{Y}$.

Finally, for $F \in \Lambda$, we let

$$\Upsilon_F = \{\mathrm{Y}_n F' \mid F \twoheadrightarrow_R F', n \geq 0\} \cup \{\mathrm{Y}_{F',F''} \mid F \twoheadrightarrow_R F', F \twoheadrightarrow_R F''\} \subset \Lambda.$$

Lemma 8. *Fix $F \in \Lambda$.*

Let $M \in \Upsilon_F$ and suppose $M \to_R M'$ for some $M' \in \Lambda$.

Then $M' = (F')^{(n)} N$ for some $F' \in \Lambda$ with $F \twoheadrightarrow_R F'$, some $N \in \Upsilon_F$, and some $n \geq 0$.

Proof. Note first that the unique one-step R-reduction of Y_n for $n \geq 0$ is Y_{n+1}, as is proven by induction on n. Next, for any $K, L \in \Lambda$, the only one-step R-reductions of $\mathrm{Y}_{K,L}$ are $K(\mathrm{Y}_{L,L})$ and $\mathrm{Y}_{K',L}$ or $\mathrm{Y}_{K,L'}$ with $K \to_R K'$ and $L \to_R L'$.

Now, let M and M' be as in the hypothesis. If $M = \mathrm{Y}_n F'$ with $F \twoheadrightarrow_R F'$, then the three possibilities for M' are $\mathrm{Y}_{n+1} F' \in \Upsilon_F$, $\mathrm{Y}_n F'' \in \Upsilon_F$ where $F' \to_R F''$, or finally $(F')^{(n)} \mathrm{Y}_{F',F'}$, where $Y_{F',F'} \in \Upsilon_F$. If $M = \mathrm{Y}_{F',F''}$, then M' is either $F' \mathrm{Y}_{F'',F''}$ or is of the form $\mathrm{Y}_{F',F'''}$ or $\mathrm{Y}_{F''',F''}$ with $F \twoheadrightarrow_R F'''$. □

It will be convenient for us to consider an enlarged set $\widetilde{V} = V \cup V'$ of variables, where V' is some set disjoint from V. In fact, we take $V' = V'_F$ to be (isomorphic to) to the set Υ_F of Definition 7 for some $F \in \Lambda$. Given $M \in \Upsilon_F$, we write v_M for the corresponding element of V'_F.

The variables in V'_F will allow us to keep track of "which variables correspond to which instances of (reductions of) $\mathrm{Y}f$" as indicated in the above outline.

Definition 9. Fix $F \in \Lambda$.

We write $\widetilde{\Lambda}_F$ for the set of lambda-terms with variables in $\widetilde{V}_F = V \cup V'_F$. For $M \in \widetilde{\Lambda}_F$, we write $\widetilde{\mathrm{FV}}_F(M) \subset \widetilde{V}_F$ for the set of free variables, and we set $\mathrm{FV}(M) = \widetilde{\mathrm{FV}}_F(M) \cap V$ and $\mathrm{FV}'_F(M) = \widetilde{\mathrm{FV}}(M) \cap V'_F$. Note that $\Lambda = \{M \in \widetilde{\Lambda}_F \mid \mathrm{FV}'_F(M) = \emptyset\} \subset \widetilde{\Lambda}_F$.

We define the *realization* map $\rho \colon \widetilde{\Lambda}_F \to \Lambda$ by substituting M for each $v_M \in V'_F$; and given $y \in V$, we define the *forgetful* or *flattening* map $\varphi_y \colon \widetilde{\Lambda}_F \to \Lambda$ by substituting y for each variable in V'_F; both ρ and φ_y are defined by recursion in an evident manner.

Lemma 10. *Fix $F \in \Lambda$, and let $M, N \in \widetilde{\Lambda}_F$. Then*

$$\rho(M[v := N]) = \rho(M)[v := \rho(N)]$$

for any $v \in V - \mathrm{FV}(F)$, and

$$\varphi_y(M[v := N]) = \varphi_y(M)[v := \varphi_y(N)]$$

for any $v \in V$ and $y \in V - \{v\}$.

Proof. By induction on M, using that $\rho(K) = \varphi_y(K) = K$ for $K \in \Lambda$, and that $K[w := L] = K$ for $K, L \in \Lambda$ and $w \in V - \mathrm{FV}(M)$. We note that the hypothesis $v \notin \mathrm{FV}(F)$ is equivalent to $v \notin \mathrm{FV}(K)$ for all $K \in \Upsilon_F$, and it is the latter which is actually used in the proof. □

We have the following variant of Lemma 5:

Lemma 11. *Fix $F \in \Lambda$. For any $M \in \widetilde{\Lambda}_F$, if $\rho(M)$ is R-reduced, then $\mathrm{FV}'_F(M) = \emptyset$ (i.e., $M = \rho(M) \in \Lambda$).*

Proof. By induction on M, using that no element of Υ_F is R-reduced. □

Now comes the crucial definition. As explained in the above outline, we will be considering a sequence of reductions of a term $\mathrm{Y}F$, and producing a parallel sequence in which each instance of $\mathrm{Y}F$ or some reduction of it is replaced by some variable y. The following definition is what lets us lift each step of the first sequence to a step in the second.

Definition 12. Fix $F \in \Lambda$.

Given $N, N' \in \Lambda$ and $M \in \widetilde{\Lambda}_F$ with $N \to_R N'$ and $\rho(M) = N$, we define a new term

$$M' = \gamma_{N,N'}(M) \in \widetilde{\Lambda}_F$$

with $\rho(M') = N'$ and $\varphi_y(M) \twoheadrightarrow_{R_{F,y}} \varphi_y(M')$ for any $y \in V - \mathrm{FV}(N) = V - \mathrm{FV}(N')$.

$$\begin{array}{ccccc} N & \xleftarrow{\ \rho\ } & M & \xrightarrow{\ \varphi_y\ } & \varphi_y M \\ \downarrow_R & & & & \twoheaddownarrow_{R_{F,y}} \\ N' & \xleftarrow{\ \rho\ } & M' & \xrightarrow{\ \varphi_y\ } & \varphi_y M' \end{array}$$

The definition of $M' = \gamma_{N,N'}(M)$ is by recursion on N:

- If $N = v \in V$, then $N' = M = v$, and we set $M' = v$; then evidently $\rho(M') = N'$ and $\varphi_y(M) = \varphi_y(M') \twoheadrightarrow_{R_{F,y}} \varphi_y(M')$.
- If $N = N_1N_2$, we consider several sub-cases:
  - (i) If $M \in V'_F$, then $N \in \Upsilon_F$, and by Lemma 8, $N' = (F')^{(n)}N_3$ with $N_3 \in \Upsilon_F$ and $F \twoheadrightarrow_R F'$. We then set $M' = (F')^{(n)}v_{N_3}$, and then have $\rho(M') = N'$ and $\varphi_y(M) = y \twoheadrightarrow_{S_{F,y}} F^{(n)}y \twoheadrightarrow_R (F')^{(n)}y = \varphi_y(M')$.
  - (ii) Otherwise, we have $M = M_1M_2$ with $\rho(M_i) = N_i$ for $i = 1, 2$.
    - (ii-a) If $N_1 = \lambda v.L$ with $v \notin \{y\} \cup \mathrm{FV}(F)$ and $N' = L[v := N_2]$, then note that we cannot have $M_1 \in V'_F$, since Υ_F contains no λ-abstraction terms. Thus, we must have $M_1 = \lambda v.K$ with $\rho(K) = L$, and we set $M' = L[v := M_2]$. We then have $\rho(M') = \rho(L)[v := \rho(M_2)] = K[v := N_2]$ by Lemma 10, and we have $\varphi_y(M) = \big(\lambda v.\varphi_y(L)\big)\big(\varphi_y(M_2)\big)$ and $\varphi_y(M') = \varphi_y(L)[v := \varphi_y(M_2)]$ again by Lemma 10, and hence $\varphi_y(M) \to_\beta \varphi_y(M')$.
    - (ii-b) Otherwise, we have $N' = N'_1N'_2$ with $N_i \to_R N'_i$ and $N_j = N'_j$ where $\{i, j\} = \{1, 2\}$. We then set $M'_i = \gamma_{N_i,N'_i}(M_i)$ and $M'_j = M_j$, and set $M' = M'_1M'_2$. We then have $\rho(M') = \rho(M'_1)\rho(M'_2) = N'_1N'_2 = N'$; and we have $\varphi_M(M_i) \twoheadrightarrow_{R_{F,y}} \varphi_y(M'_i)$ for $i = 1, 2$ and hence $\varphi_y(M) \twoheadrightarrow_{R_{F,y}} \varphi_y(M')$.
  - (iii) If $N = \lambda v.N_1$ with $v \notin \{y\} \cup \mathrm{FV}(F)$, then $M = \lambda v.M_1$ where $\rho(M_1) = N_1$. We again consider sub-cases:
    - (iii-a) If $N_1 = N_2v$ for some term N_2, and $N' = N_2$ (i.e., if $R = \beta\eta$ and $N_1 \to_\eta N_2$), then $M_1 = M_2v$ where $\rho(M_2) = N_2$, and we set $M' = M_2$. We then have $\rho(M') = N'$ and $\varphi_y(M) = \lambda v.\varphi_y(M_2)v \to_\eta \varphi_y(M')$.
    - (iii-b) Otherwise, $N' = \lambda v.N'_1$ with $N_1 \to_R N'_1$. We then set $M'_1 = \gamma_{N_1,N'_1}(M_1)$ and set $M' = \lambda v.M'_1$, and we have $\rho(M') = \lambda v.\rho(M'_1) = \lambda v.N'_1 = N'$ and $\varphi_y(M) = \lambda v.\varphi_y(M_1) \twoheadrightarrow_{R_{F,y}} \lambda v.\varphi_y(M'_1) = \varphi_y(M')$.

(*End of definition of* $\gamma_{N,N'}$.)

The following proposition (applied to $L = yN$), together with Proposition 6, immediately implies the theorem.

Proposition 13. *Let* $F, L \in \Lambda$ *and* $y \in V - \mathrm{FV}(F)$*. If* $L[z := \mathrm{Y}F]$ *has an* R*-normal form, then* L *has the same* $R_{F,y}$*-normal form.*

Proof. Let L be a term such that $N_0 = L[y := \mathrm{Y}F]$ has an R-normal form. We thus have a sequence $N_0, \ldots, N_n$ where N_n is R-reduced and $N_i \to_R N_{i+1}$ for all $0 \leq i < n$.

$$
\begin{array}{ccccc}
N_0 & \to_R & \cdots & \to_R & N_n \\
 & & \uparrow{\scriptstyle \rho} & & \\
M_0, & & \cdots, & & M_n \\
 & & \downarrow{\scriptstyle \varphi_z} & & \\
L = \varphi_z(M_0) & \twoheadrightarrow_{R_{F,y}} & \cdots & \twoheadrightarrow_{R_{F,y}} & \varphi_y(M_n)
\end{array}
$$

FIGURE 1. The behaviour of the M_i's and N_i's in Proposition 13.

Define $M_0 = L[y := \upsilon_{\mathrm{Y}F}] \in \widetilde{\Lambda}_F$, so that $\rho(M_0) = N_0$ and $\varphi_y(M_0) = L$. Now for each $0 \leq i < n$, define $M_{i+1} = \gamma_{N_i,N_{i+1}}(M_i)$; we then have $\rho(M_{i+1}) = N_{i+1}$ and $\varphi_y(M_i) \twoheadrightarrow_{R_{F,y}} \varphi_y(M_{i+1})$, and in particular, $L \twoheadrightarrow_{R_{F,y}} \varphi_y(M_n)$ (see Figure 1).

Since N_n is R-reduced, $\mathrm{FV}'_F(M_n) = \emptyset$ by Lemma 11, hence $N_n = M_n = \varphi_y(M_n)$ and thus $L \approx_{R_{F,y}} N_n$. Since N_n is R-reduced and $y \notin \mathrm{FV}(N_n)$ (this follows from $N_n \approx_R N_0 = L[y := \mathrm{Y}F]$ and $y \notin \mathrm{FV}(\mathrm{Y}F)$), it is also $R_{F,y}$-reduced, as desired. □

4. OTHER FIXED POINT COMBINATORS

There are (infinitely many) other terms M in the λ-calculus which are fixed-point combinators in the sense that $F(TF) \approx_R TF$ for all $F \in \Lambda$. A well-known example is Turing's combinator $\Theta = (\lambda x.\lambda y.y(xxy))(\lambda x.\lambda y.y(xxy))$; see [Bar84, Definition 6.1.4].

In the case of Θ, it is easy to adapt the above proof to see that it is also a least fixed point combinator. The main point is to modify Definition 7 in light of the possible β-reductions of ΘF: the set Υ_F should now consist of all terms $\Theta' F'$ with $\Theta \twoheadrightarrow_R \Theta'$ and $F \twoheadrightarrow_R F'$. The statement of Lemma 8 then still holds, the construction in Definition 12 goes through verbatim, and the statement and proof of Proposition 13 go through *mutatis mutandis*, substituting Θ for Y.

The proof could similarly be adapted for other fixed-point combinators; it is just a matter of modifying Definition 7 so that Lemma 8 remains true. In fact, in light of [Bar84, Theorem 19.3.4], it is plausible that *every* fixed point combinator is a least fixed point combinator. Moreover, given the result of [Gol05] giving a recursive enumeration of all fixed point combinators, proving this conjecture is perhaps not out of reach.

REFERENCES

[Bar84] H. P. Barendregt. *The lambda calculus. Its syntax and semantics.* Revised. Vol. 103. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1984, pp. xv+621.

[Gol05] Mayer Goldberg. "On the Recursive Enumerability of Fixed-Point Combinators". In: *BRICS Report Series* 12.1 (2005).