A CHARACTERIZATION OF THE SIMPLICIAL CATEGORY

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1. Introduction

This note concerns a certain seeming *ad-hoc*ness in the definition of simplicial sets which irked me for a long time.

When defining simplicial sets, we make use of the category $\mathbf{\Delta}$ of non-empty finite ordinals with non-decreasing functions. Of course, as the name ($\mathbf{\Delta}$) suggests, the category has a certain geometric interpretation, which is what originally motivated it.

The usual embedding $\mathbf{\Delta} \to \mathbf{Top}$ makes this geometric interpretation explicit. Considering first the subcategory $\mathbf{\Delta}_+$ having only the strictly increasing maps (which is used in the definition of "semi-simplicial" sets), we note that it is isomorphic to the subcategory of **Top** whose objects are the standard n-simplex Δ^n for each n, and whose maps are the inclusions of the faces (of all dimensions) into each simplex. Of course, in order to make sense of this, we must choose an identification of Δ^k with each k-dimensional face of each Δ^n.

One way to do this is to just order the vertices of each simplex, and take only the maps which are order-preserving on the vertices, which makes the isomorphism to $\mathbf{\Delta}_+$ immediate.

Now, we might imagine picking some other identifications of lower-dimensional simplices with the faces of higher-dimensional ones. However, we cannot do this arbitrarily. For one, the chosen inclusions might not be closed under composition (as one can easily verify with an example). Of course, we could avoid this simply by taking the category generated by all the chosen inclusions, but then the resulting category would have the undesirable property of having *more than one* inclusion of some k-simplex into the same face of some n-simplex, in contradiction with the original intention that the arrow should be the inclusions of faces.

However, if we demand that the inclusions be closed under composition, this determines the category:

Proposition 1. *Let* $\mathbf{C}$ *be a subcategory of* **Top** *whose objects are the* Δ^n $(n \geq 0)$ *and whose morphisms are simplicial injections, such that for each* $k < n$*, there is exactly one morphism from* Δ^k *onto each* k*-dimensional face of* Δ^n*. Then* $\mathbf{C}$ *is isomorphic to* $\mathbf{\Delta}_+$*.*

Remark. Actually, in the proof, we will only use the weaker assumption that for $k = 1$, there is exactly one morphism from Δ^k onto each edge of Δ^n, and for $k \neq 1$, there is *at least* one morphism from Δ^k onto each k-dimensional face of Δ^n.

For the case of $\mathbf{\Delta}$, it is of course isomorphic to the subcategory of **Top** whose objects are the Δ^n and whose arrows are the linear maps which are non-decreasing on the vertices (with respect to some orderings). Here, the intention is that there should be exactly one map from Δ^k to each "possibly degenerate k-dimensional face" of Δ^n. This means that for each partition of k over the vertices of Δ^n, there should be exactly one map for which the number of vertices of Δ^n mapping to each vertex of Δ^k is the corresponding number. Again, this property characterizes the category:

Proposition 2. *Let* $\mathbf{C}$ *be a subcategory of* **Top** *whose objects are* Δ^n $(n \geq 1)$ *and whose morphisms are simplicial maps and having the following property: for each* k *and* n *and each partition of* k *over the vertices of* Δ^n*, there is exactly one morphism* $\Delta^k \to \Delta^n$ *such that the cardinalities of the preimages of the vertices are the given partition. Then* $\mathbf{C}$ *is isomorphic to* $\boldsymbol{\Delta}$*.*

Remark. Again, we need actually only assume that there is at least one morphism corresponding to each partition of k over the vertices of Δ^n, and that there is exactly one map from Δ^1 onto each edge of Δ^n.

Let us now prove these claims.

2. The proofs

Proof of Proposition 1. Let $\mathbf{C}$ be as in the statement of the theorem. We will find orderings of the vertices of each Δ^m making all of the morphisms of $\mathbf{C}$ order-preserving.

For Δ^0 there is nothing to do and for Δ^1 we choose any ordering.

Next, note that each edge of each Δ^m has an orientation, given by the unique morphism from Δ^1 onto that edge. Moreover, these orientations are preserved by the morphisms of $\mathbf{C}$. Hence, it remains to see that the orientations induce a total ordering on the vertices of Δ^m – i.e., that there are no oriented cycles.

We first consider the case $m = 2$. If the edges of Δ^2 formed an oriented cycle, then each face of Δ^3 would have to as well (since the inclusions are orientation-preserving) which is clearly impossible (draw a picture).

Now assume $m > 2$. We cannot have any 3-cycles in Δ^m, as this would give rise to a non orientation-preserving map from Δ^2. So assume that Δ^m has a cycle $v_0 \to v_1 \to \cdots \to v_i \to v_0$ with $i > 3$. Since there are no 3-cycles, and since we have $v_0 \to v_1 \to v_2$, we must have $v_0 \to v_2$. Similarly, by induction, we have $v_0 \to v_j$ for each $j \leq i$. But this contradicts the assumption $v_i \to v_0$. □

Proof of Proposition 2. Let $\mathbf{C}$ be as in the statement of the theorem.

Note that the subcategory of $\mathbf{C}$ containing all the injective morphisms satisfies the hypothesis of Proposition 1, hence the injective morphisms are all order-preserving with respect to some orderings on the vertices of each Δ^n. It now remains only to show that all the morphisms of $\mathbf{C}$ are order-preserving.

But this is clear, since if there were a non-order-preserving morphism $\Delta^k \to \Delta^n$, we could compose it with a morphism $\Delta^1 \to \Delta^k$ to obtain an injective non-order-preserving morphism $\Delta^1 \to \Delta^n$. □